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ABSTRACT

The materials collected here were presented at the fourth annual meeting held October 23 through 25, 1982, at the University of Georgia. The papers are grouped under the following headings: Mathematical Abilities; Understanding; Early Number; Adolescent Reasoning; Problem Solving; Teaching and Teacher Education; and Technology. Space limitations required that the papers be short, but each item is viewed to be state-of-the-art work in its field. The document concludes with an author index, mailing addresses, and an addendum consisting of three additional papers. (MP)

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PROCEEDINGS OF THE FOURTH ANNUAL MEETING
OF THE NORTH AMERICAN CHAPTER
OF THE INTERNATIONAL GROUP
FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

ATHENS, GEORGIA

OCTOBER 23-25, 1982

EDITED BY: SIGRID WAGNER

DEPARTMENT OF MATHEMATICS EDUCATION
UNIVERSITY OF GEORGIA
ATHENS, GEORGIA 30602

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PREFACE

The International Group for the Psychology of Mathematics Education (PME) was founded in 1976 at the Third International Congress for Mathematics Education. The group sponsors an annual research conference to stimulate the exchange of ideas among researchers who are interested in the psychology of learning and teaching mathematics.

Interest in the group has grown so rapidly that a North American Chapter (PME-NA) was founded in 1979. Like its parent organization, PME-NA also sponsors an annual research conference. The papers contained herein were presented at the Fourth Annual Meeting held October 23-25, 1982, at the University of Georgia.

The papers are grouped under seven headings:

A. Mathematical Abilities
B. Understanding
C. Early Number
D. Adolescent Reasoning
E. Problem Solving
F. Teaching and Teacher Education
G. Technology

Space limitations require that the papers be short, but each one reports state-of-the-art work in its field. Authors' mailing addresses are included in the index for the convenience of readers who may want more detailed information than can be presented here.

In addition to the contributed papers, three invited lecturers spoke at general sessions of the conference. Dr. Kathleen M. Hart of Chelsea College, University of London, presented the opening address, entitled "I Know What I Believe; Do I Believe What I Know? Communications with Teachers and Children." Dr. Larry Hatfield, University of Georgia, talked about "Instructional Computing in Learning and Teaching Mathematics." Dr. Stefan Turnau, Belmont Abbey College, discussed "Mathematics Education Research in Poland - Some Recent Issues and Results." It is anticipated that the essence of each of these presentations will eventually be published elsewhere, either as journal articles or as part of a book.

The editor was gratified at the number of excellent papers submitted for the conference. She thanks the authors for their cooperation in sending their manuscripts on time and in the form requested. She also thanks the Department of Mathematics Education at the University of Georgia for their cooperation, support, and encouragement. She especially thanks her secretary, Rosemary Banks, for her cheerful, capable, and conscientious help during each phase of preparation for the conference.

S. Wagner

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MATHEMATICAL ABILITIES: THE VIEW FROM FACTOR ANALYSIS

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A pool of factor-analytic studies of mathematical abilities was collected. The conceptual comparison of the results of the selected studies is organized into families of factors. The General family is relatively unexplored in the pool. Abilities represented by factors in the Numerical family seem to be basic to mathematics achievement but not integral to it. For the Reasoning family, the results in the selected studies confirm the close correlation between reasoning and mathematical abilities but do little to clarify the nature and details of the relationship. The abilities represented by factors in the Spatial family seem to play a role in mathematics achievement, but only in certain aspects of it. The Verbal family seems to have minimal influence on mathematics achievement. The Mathematical family represents specific learning in mathematics. The existence of factors in the family opens the possibility of abilities that are specific to mathematics.

The structure of mathematical abilities, as studied through factor analysis, is well represented within the literature on mathematical abilities. In the present examination, 48 such studies (hereafter referred to as the pool of studies, see Note 1) were selected and a conceptual comparison of results was undertaken. The specific questions for this part of the study (see Note 2) were: a) How are the results of the selected studies similar and how are they different? b) What factors appear to be similar across the studies? c) What mathematical variables load on these factors?

PROCEDURES

Five families of factors were formed and operational definitions developed for each: General, Numerical, Reasoning, Spatial, Verbal, and Mathematical. A family of factors represents a class of abilities, possibly very broad and very divergent but in some way associated with each other. No claim is made that each family represents a single ability or even a small number of closely related abilities. Then, each factor in each study was classified in one of the families (see Note 1). The classifications were based on the variables that loaded on the factor, the original author's interpretation of the factor, and the present author's interpretation of the factor. The procedures for the examination of each family were neither set nor uniform. Perhaps the underlying method could be described as "vote-counting", with the design and significance of each study being considered in the weight of its "vote."

THE GENERAL FAMILY

The General family of factors consists of two kinds of general factors: a) the large, general factor extracted first in an analysis; and b) general factors extracted in a second-order analysis. Though the kinds were arrived at by quite different methods of analysis, they are relatively similar in the abilities they represent. They commonly represent abilities to discover and use relationships in a variety of tasks and are some combination of fluid and crystallized intelligence. They are quite narrow compared to their potential scope, as illustrated in Cattell's (1971; Hakstian & Cattell, 1978) triadic theory of ability structure. However, the results do not imply that other general functions (e.g., Visualization Capacity or General Retrieval Capacity) play no role in mathematical abilities. Rather, their role has not been explored in the studies.

Since the General family of factors has been relatively unexplored in the studies, its role in mathematics achievement is also relatively unexplored. What attention is given to this role shows general factors to be closely related to mathematics achievement. Such a relationship is intuitively plausible when one recognizes the emphasis on reasoning in many of the variables used to define general intelligence.

THE NUMERICAL FAMILY

The Numerical family of factors consists of the standard Number factor and of other computational factors that include reasoning or algebraic skills. It represents speed and facility in manipulating numbers and other symbols according to simple or well-practiced rules, particularly if the rules originally involved reasoning.

Thirty-nine of the selected studies identified one or more numerical factors. Many of these factors have varying degrees of correspondence to the standard Number factor. That is, they are identified by variables measured by highly speeded and simple computational tasks. Yet, there are also important differences among the factors. The greatest source of differences is the presence or absence of significant loadings by reasoning variables. The amount of reasoning decreases with age until, in the studies with older subjects, the numerical factors involve few or no reasoning variables. Werdelin's (1958) automatization process can be used to explain the increasing clarity of the factors. Automatization is a process by which performance on tasks becomes automatic, without conscious effort. So, as facts and skills are practiced more, they become clearer indicators of individual differences in automatization. Further, under certain circumstances, tasks other than the basic operations with whole numbers might become automatized and load solely on the Number factor or define a separate factor. Several investigators found more than one numerical factor; in every case, one factor could be identified with the standard Number factor, and the others involved variables with complex operations and, sometimes, reasoning variables. The presence of reasoning variables suggests that these numerical factors might be characterized by less complete automatization than the Number factor. Also, there is clear evidence in studies in the pool that certain algebraic skills are automatizable. It is not clear to what extent they can be or which ones can be automatized.

Many studies, both within and outside the present pool, deal with the relationship between numerical abilities and mathematics achievement. In general, though numerical abilities initially relate to achievement in school mathematics, their influence declines as the student progresses in the mathematics curriculum. Thus, numerical abilities seem to be basic to mathematics achievement but not integral to it. In other words, a certain minimal level of numerical ability seems essential to achievement in mathematics, but ability beyond this level may be irrelevant to such achievement.

THE REASONING FAMILY

The Reasoning family consists of a wide variety of factors. It includes such standard factors as Induction, Deduction, General Reasoning, Judgment, and Integration. General Reasoning is referred to as Arithmetic Reasoning here because it is identified by variables that measure the subject's ability to solve arithmetic word problems. Within the studies in the pool, the family also includes factors with such names as Abstract Reasoning, Implicit Reasoning, and Approach-to-problem-solving. All represent abilities to reach or evaluate conclusions about a situation or a set of data, based on given or assumed information.

In all but nine studies in the pool, the researchers identified at least one reasoning factor, yielding a total of 145 factors, substantially more than in any of the other families. Clearly, the researchers believed reasoning abilities to be correlated with mathematical abilities, and in some cases even equivalent to mathematical abilities. Yet, the same inconsistencies in reasoning factors that are seen in the general factor-analytic literature are also found in the Reasoning family within the pool. The present examination does not satisfactorily simplify the Reasoning family. It serves mainly to confirm the close correlation between reasoning and mathematical abilities and does little to clarify the nature and details of the relationship.

Fifty of the reasoning factors have loadings by mathematics achievement variables other than Arithmetic Reasoning variables. The most striking feature of these mathematics achievement variables is their variety and scope. Almost every kind of mathematics variable in the pool loads on some reasoning factor--variables that represent computation, understanding, and routine problem solving in arithmetic, algebra, and geometry. The results of the studies in the pool strongly support a close association between reasoning abilities and mathematics achievement. However, further explorations of the relationship were not fruitful.

THE SPATIAL FAMILY

The Spatial family of factors includes the standard factors of Spatial Orientation, Visualization, Spatial Relations, and Flexibility of Closure. It involves the abilities to perceive, understand, manipulate, and relate spatial material. Most of the spatial factors in the selected studies are defined by variables that identify more than one of the standard factors. One or more spatial factors were identified in 22 studies in the pool, for a total of 35 spatial factors. No two of the spatial factors can be said to be identical, even factors found by the same researcher. And in many of the studies, the spatial factor is merely identified as a spatial factor, with little or no further interpretation, or it is related to a well-known spatial factor.

It was widely believed by the investigators represented in the pool that spatial abilities play a role in mathematics achievement. Some investigators believed the role was so obvious that they reiterated its importance even though the mathematical variables they included did not load on their spatial factor. The evidence from the pool suggests that spatial abilities play a role in mathematics achievement, but only in certain aspects of it. There is no evidence to establish a relationship between spatial abilities and arithmetic achievement or between spatial abilities and algebraic achievement. However, there is evidence to both establish and support a relationship betweeen spatial abilities and geometric achievement. Werdelin's (1961) results clearly establish a relationship between spatial abilities and certain geometric tasks. Other results within the pool are not inconsistent with his results and, in some cases, support them. However, as several studies suggested, other factors such as general intelligence are more important to geometric achievement than spatial abilities.

THE VERBAL FAMILY

The Verbal family of factors consists primarily of such standard factors as Word Knowledge and Reading Comprehension and secondarily of fluency factors. It represents abilities to comprehend language in single words, sentences, or paragraphs (verbal factors) and to produce words with certain restricting conditions (fluency factors). Overall, verbal abilities seem to have minimal influence on mathematics achievement once basic reading abilities have been developed. The influence of fluency factors on mathematics achievement has not been explored.

THE MATHEMATICAL FAMILY

The Mathematical family consists of factors that represent specific learning in mathematics or some part of mathematics. The factors can be considered agencies in Cattell's (1971) triadic theory of ability structure. The boundaries for this family were difficult to set because they are essentially the distinction between ability and achievement. Many psychometricians have abandoned any attempt to distinguish these concepts. However, the formation of the family provided a frameword in which to examine the question of the generality or specificity of mathematical abilities. That is, are all mathematical abilities general abilities applied to mathematics, or are some mathematical abilities specific to mathematics?

In the selected studies, mathematical variables have defined mathematical factors, regardless of the sample or variables in the battery, even though specific mathematical factors have not been identified consistently across studies. Thus, mathematical factors have been clearly established as a kind of factor found in the pool. Does their existence imply the existence of specific mathematical abilities? Clearly, the evidence in the pool shows that mathematics achievement depends upon a variety of factors that are not specific to mathematics. But the existence of mathematical factors also opens up the possibility of specific mathematical abilities. Such abilities are only one possible source that could produce such factors. Personality variables, teacher variables, socio-economic variables, learning style variables, and many others are likely to affect such factors. It is beyond the scope of factor analysis, as used in the studies in the pool, to determine one source out of many for the individual differences that generate a factor. However, even though the mathematical factors do not prove or disprove the existence of specific, mathematical abilities, they do clearly show that not all variance in mathematics achievement is accounted for by general abilities.

MISCELLANEOUS FACTORS

In twelve studies in the pool, factors were identified that did not fit into any of the families. Most of the miscellaneous factors relate to areas that have been widely explored in the general factor-analytic literature but that have not been explored widely in the selected studies. It would be inappropriate to conclude that they are unrelated to mathematical abilities. Rather, the relationships have not been adequately explored.

CONCLUSION

Such is the view of mathematical abilities, as seen from factor analysis. The view is a narrow and incomplete one. Only studies that met certain criteria were included in the examination; studies that investigated mathematical abilities through non-factor-analytic statistical techniques or through clinical designs were not included. Also, the examination provides only a conceptual comparison of the results in the selected studies, which future researchers in mathematical abilities may find useful in guiding further research. Yet, at times, a narrow view of a subject can provide greater clarity to the part of it being viewed. Hopefully, this examination has provided some clarity to the view of mathematical abilities, as seen from factor analysis.

REFERENCE NOTES

1. Space does not permit listing all the studies in the pool or reproducing the chart of factors in the families. These may be obtained from the author.
2. The complete study also includes reanalyses of twelve data sets in the pool and a comparison across the reanalyzed results. It is the author's dissertation, to be filed at the University of Georgia, 1982.

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GENERALITY OF THE RELATIONSHIP BETWEEN SPATIAL VISUALIZATION AND MATHEMATICAL REASONING

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This study was designed to examine: 1) whether the relationship between mathematical reasoning and spatial ability was affected by socio-economic-status (SES), and 2) the strength of the relationship after general intellectual ability was statistically controlled. Two-hundred-sixty eighth grade students from three different schools, each selected for SES and academic diversity, participated as subjects in the study. The results found no site interaction and that the amount of variance common to spatial ability and mathematical reasoning was small (7%) after controlling for general intelligence.

A relationship between spatial visualization, or spatial ability, and mathematical reasoning has been hypothesized for some time, with considerable support (McGee, 1979; Smith, 1964). This relationship has also been posited to account for at least part of the gender difference in mathematical reasoning (McGee, 1979). Some researchers have recognized the possible educational significance of spatial ability for teaching mathematical reasoning and have attempted to increase mathematical reasoning and/or reduce gender differences in mathematical reasoning by increasing spatial ability (e.g., Cook & Kersh, 1980). However, the nature of the relationship between mathematical reasoning and spatial ability is not well understood, particularly its generalizability and uniqueness.

Most of the research on the relationship between mathematical reasoning and spatial ability has not investigated the possibility of an interaction with socioeconomic status (SES) (Pulos, Stage & Karplus, 1982). Other research (e.g., Pulos, Stage, Karplus & Karplus, 1981) suggests that most of the relationship between spatial ability and mathematical reasoning is moderated by other cognitive abilities in a middle class population. This finding may not be replicated in other populations, however. The purpose of the current paper is to investigate these two issues. Specifically, the current study was conducted to answer the following questions:

1. Are the relationships among gender, spatial ability, and mathematical reasoning affected by SES?
2. How much of the common variance between spatial ability and mathematical reasoning is due to general intellectual ability?

METHOD

Subjects and Settings

The subjects were 260 eighth grade students from three urban schools within two miles of one another. They were randomly selected from the schools, excluding those students in remedial mathematics classes two or more years below grade level.

The schools, designated A, B, and C in descending order of relative SES, were chosen for their academic and ethnic diversity. Minority enrollment of 82% to 92% consisted primarily of Asians, Blacks, and Hispanics.

School A. School A, which contributed 57 males and 45 females is located in a relatively desirable residential location. District records showed that only 16% of the families served by the school received Aid to Families with Dependent Children (AFDC) funds, an indicator of relative affluence in a district where the average AFDC rate was 39%. In Spring, 1981, School A averages were at the 70th percentile of the national norms on the mathematics subtest of the Comprehensive Test of Basic Skills (CTBS). The teachers supplemented the texts with materials that challenged the students "to think" and attended inservice programs that emphasized problem solving.

School B. School B is located closest to the city center and 12% of the families received AFDC support. It contributed 44 males and 39 females. School B students scored at the 65th percentile in mathematics on the CTBS. The mathematics faculty were experienced, though somewhat less likely to attend inservice training.

School C. School C, where 42 males and 33 females participated, is in one of the older sections of the city and 43% of the families received AFDC support. The school's students scored at the 31st percentile on mathematics on the CTBS. The mathematics faculty had little preservice experience in mathematics (none had degrees in mathematics), and less inservice education than teachers at the other schools.

Procedure and Material

The interviewers represented each of the ethnic groups in the sample and some were local residents. Approximately half the students were tested by members of their own ethnic group.

Mathematical reasoning tasks. Mathematical reasoning was assessed with two different types of problems, proportions and reasoning with unknowns. After two especially simple problems established the subject's understanding of the procedure, six items of each type were presented to each student in an individually administered interview. To minimize memory demands, tasks were presented orally and in written or diagram form, and paper and pencil were provided. Assistance was given with computation if requested by the student.

Proportional reasoning was assessed by the Lemonade Puzzles (Pulos et al., 1982) in which students are presented with pictorial and verbal descriptions of two different mixtures of sugar and lemon juice and asked to compare the tastes, judge which recipe is sweeter, and make the recipes taste the same when unequal.

The reasoning with unknowns task used the Number Puzzles (Pulos et al., 1981) in which students were asked to solve such questions as, "I am thinking of a number. I multiply my number by 4 and subtract 12. I get 32. What is my number?"

Spatial reasoning. Three spatial reasoning tasks were employed: paper folding, embedded figures, and the Water Level Task. The paper folding task was a form of the ETS paper folding task adapted for individual administration (Pulos et al., 1981). The subject's goal was to anticipate the appearance of a folded and punched piece of paper after it was unfolded. The embedded figure test was a modified form of the Group Embedded Figures Test modified to reduce the memory demand of the task. The subject's goal was to find a simple shape embedded in a more complex shape. The third task was the Water Level Task in which the subject's goal was to draw a line corresponding to the imagined surface of water in a tilted bottle.

Cognitive correlates. Two group-administered measures of general intellectual ability were administered to all subjects -- the vocabulary and series completion sections from the Primary Mental Ability Test. These tests represent two different facets of intelligence (Horn, 1978), crystallized intelligence and fluid intelligence, respectively.

Analysis. Multiple measures of both mathematical reasoning and spatial visualization were employed to accentuate the common aspects across tasks and minimize the unique aspect of each task (Humphreys, 1974). A single score was computed for each ability in which the constituent tasks were equally weighted. The use of these single scores also minimized the problem of multicollinearity which would have emerged if multiple scores were used.

The primary mode of analysis was a hierarchical multiple regression, with mathematical ability as the dependent variables and the independent variables in the following order: crystallized intelligence, fluid ability, spatial ability, school, gender, and the interaction terms for school X spatial ability, gender X spatial ability, gender X school X spatial ability. The variables were organized hierarchically for the following reasons. Performance on the vocabulary test depends only slightly on factors measured by the other tasks, so it is entered first. The series completion task probably depends on crystallized intelligence but little else, so it is second. The spatial tasks depend on both of these (Horn, 1978), and the space score is thus third. Gender and school are both packaged variables in which part of the variance is probably due to the cognitive variables so they were entered later, with school as a dummy variable. The interaction terms were entered stepwise after the main effects (Cohen & Cohen, 1975).

RESULTS AND CONCLUSION

The results of the multiple regression are summarized in Table 1. Only variables having a statistically significant relationship, beyond the .05 level, are included in the table. The variables not reported were not only not significant but none of them ever accounted for even 1% of the variance. Table 1 indicates that the only significant predictors of mathematical reasoning were the intelligence scales and spatial ability. None of the interaction terms was significant. However, while spatial ability was significant, the amount of variance it explained in mathematical reasoning was slight after the common relationship with intelligence was removed, changing from 25% to 7% of the variance.

Table 1

Summary of Multiple Regression on Mathematical Reasoning

Variable	F	R	r
Vocabulary	47.80***	.40	.40***
Series Completion	58.61***	.57	.56***
Spatial Visualization	25.29***	.62	.49***

*** = $p < .001$

The results of the multiple regression do not appear to be affected by site or gender. Neither site nor gender nor any interactions were signicant. However, when the mathematical reasoning and spatial ability were considered separately, apart from the cognitive correlates, an ANOVA found a very significant effect of site on both mathematical and spatial ability ($F = 25.686$, $p < .001$; $F = 14.73$, $p < .001$; respectively) and a significant effect for gender on spatial ability only ($F = 5.84$, $p < .05$).

In summary, it appears that spatial ability is related to mathematical reasoning in a general, but primarily indirect manner, which may be of only marginal educational significance. The relationship is fairly robust, not being effected by location. However, most of the relationship observed between spatial ability and mathematical reasoning is due to a common relationship with intelligence. Thus it seems unlikely that training on spatial ability would have much of a general effect on mathematical reasoning. Intervention should, instead, focus on teaching general reasoning and problem solving skills.

The study also suggests that the relationship between gender and mathematics appears to be different from the relationship between gender and spatial ability. While a significant simple correlation between gender and spatial ability was found, as was a relationship between mathematical reasoning and spatial ability, no relationship between gender and mathematical reasoning was obtained. Thus, it appears that intervention for ameliorating gender differences in mathematics, when they are found, based upon improving spatial ability may not be the most productive intervention.

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THE SPATIAL VISUALIZATION UNIT AND THE FACTORS AND MULTIPLES UNIT OF THE MIDDLE GRADES MATHEMATICS PROJECT--IMPLEMENTATION AND RESULTS

David Ben-Haim, William Fitzgerald, and Glenda Lappan
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The Spatial Visualization Unit was implemented in grades 6-8. Prior to instruction, there were significant grade differences in Spatial Visualization performance (increasing with age) and sex differences (favoring boys). After instruction there was significant gain for all grade levels and in spite of initial sex differences, boys and girls made comparable gains. The Factors and Multiples Unit was implemented in grades 5-8. The results of analyses of pre-post test scores showed significant growth for all grade levels and no sex differences.

The MGMP is a curriculum development project funded by NSF - DISE, to develop units of high quality mathematics instruction for grades 5 through 8. Each unit:

- is based on a related collection of important mathematical ideas,
- provides a carefully sequenced set of activities which lead to an understanding of the mathematical challenges,
- helps the teacher foster a problem solving atmosphere in the classroom,
- uses concrete manipulatives, where appropriate to help provide the transition from concrete to abstract thinking,
- utilizes an instructional model which consists of three phases...launching, exploring, and summarizing,
- provides a carefully developed instructional guide for the teacher,
- requires two to three weeks of instructional time.

The goal of the MGMP materials is to help students develop a deep, lasting understanding of the mathematical concepts and strategies studied. Rather than attempting to break the curriculum into small bits to be learned in isolation from each other, MGMP materials concentrate on a cluster of important ideas and the relationships which exist among these ideas. Where possible the ideas are embedded in concrete models to assist the students in moving from this concrete stage to more abstract reasoning.

Two units, Spatial Visualization and Factors and Multiples, are complete, have been classroom tested and evaluated. Two more units, Probability and Similarity are currently under development.

SPATIAL VISUALIZATION

Spatial Visualization is considered to be one of the two major factors of spatial ability. Spatial Visualization is an aptitude that deals with the mental manipulation of rigid figures. Investigating spatial visualization is important because of its relationship to most technical-scientific occupations (Harris, 1981) and especially to study of mathematics, science, art, and engineering (Bishop, 1978; Harris, 1981). The spatial visualization aptitude emerges as a component of mathematics ability in most factor analytic studies (Schonberger, 1976), and it shows about the same magnitude of correlation with mathatics achievement as the verbal component does (Fennema & Sherman, 1977). An additional indication that this aptitude is an important consideration is the finding of sex-related differences in spatial visualization. Surprisingly, studies of training programs to increase spatial visualization are very few. Of the existing studies, most have been done with adults and the results are inconclusive. No large-scale study to implement a unit of instruction in spatial visualization for a wide range of grade levels was found. This provided motivation for the study of spatial visualization in grades five through eight reported below (Ben-Haim, 1982).

Purpose

The study had two related purposes. The first was to determine existing differences in spatial visualization abilities and in attitudes toward mathematics of fifth through eighth grade students by sex and grade level prior to an instructional intervention. Research on cognitive variables shows close relationship of the cognitive and affective components, thus investigating performance and attitudes simultaneously seems appropriate. The second purpose was to analyze the effects of instruction in activities involving spatial visualization tasks on the skills and attitudes toward mathematics of a sample of sixth, seventh, and eighth grade students by sex and grade level.

METHODOLOGY

The study was conducted during winter 1982. Data was collected from January 20 to April 20 and included general information on the subjects and pre-post-and retention test scores on spatial visualization tests and attitude scales. The subjects of the study attended schools in a rural suburb near Lansing, the state capital of Michigan. Subjects consisted of all students in the one middle school (6th through 8th grade) in that area and all the fifth graders from two out of four elementary schools which feed the middle school. The schools serve a cross section of middle class families and are predominately white. There were 681 fifth through eighth grade students who participated in the assessment of differences prior to the instruction. Of these 430 sixth, seventh, and eighth graders participated in the evaluation of the effects of the instruction; of these, 238 took part in the evaluation of the persistence of the effects (four weeks after the end of the instruction).

The spatial visualization instructional material includes ten carefully sequenced activities. The activities involve representing three-dimensional objects (buildings made from cubes) in two-dimensional drawings and vice versa, constructing three-dimensional objects from their two-dimensional representations. Two different representation schemes are used for the two dimensional drawings. First an "architectural" scheme involving three flat views of the building--base, front view and right view. After students are comfortable with this scheme, they are introduced to isometric dot paper and hence to a representation consisting of a drawing of what one sees looking at a building from a corner. Similarities, differences, strengths and inadequacies of the two schemes are explored.

The instruments used included a semantic differential attitudes scale (Shumway et al, 1981) and a spatial visualization performance test designed by the project staff. The math teachers who taught the units were provided with the instructional material, a teacher guide and a two-hour workshop. The tests and attitude scales were administered during the regular school day by the classroom teachers. The statistical analyses included multivariate and univariate analysis of variance and repeated measures.

RESULTS AND CONCLUSIONS

Table 1 provides means of the spatial visualization test (SVT, range 0-32) and the Mathematics Attitude Scale (MAT, range 1-5) pretest scores for the entire sample by grade and by sex. Table 2 includes the P values resulted from the multivariate and univariate analysis of variance. Three planned comparisons were used for the grade main effects--(5,6) vs. (7,8), 5 vs. 6, and 7 vs. 8.

Table 1

Pretest Means of SVT and MAT Scores by Grade and by Sex

		SVT	MAT	Boys			Girls		
					SVT	MAT		SVT	MAT
	N	M	M	N	M	M	N	M	M
Grade 5	94	8.81	3.685	48	9.58	3.788	46	8.00	3.577
Grade 6	208	10.17	3.044	104	11.31	3.064	104	9.03	3.024
Grade 7	170	11.17	2.858	90	11.76	2.696	80	10.50	3.061
Grade 8	209	12.97	2.802	96	15.27	2.836	113	11.02	2.774
Total	681	11.10	3.076	338	12.33	3.007	343	9.89	3.025

Table 2

Multivariate and Univariate Analysis of Variance--P Values for Grade Planned Comparisons, Sex Differences, and Interactions

Source of	Multivarite	Univariate	
		SVT	MAT
Variation	P <	P<	P<
G1 = (5,6) vs. (7,8)	.0001	.0001	.0001
G2 = 5 vs. 6	.0001	.0235	.0001
G3 = 7 vs. 8	.0012	.0006	.4879
Boys vs. Girls	.0001	.0001	.6897
G1 by Sex	.1170	.3295	.1007
G2 by Sex	.5717	.5126	.4776
G3 by Sex	.0044	.0060	.0236

It was concluded from the analysis of the data gathered prior to the instructional intervention that there were significant (1) grade difference in spatial visualization performance (increasing with age) and in attitudes toward mathematics (decreasing with age). (2) Sex differences in spatial visualization performance (favoring boys), but no sex differences in attitudes toward mathematics. In addition, the data indicated that the sex differences in spatial visualization skills increased significantly from grade seven to grade eight.

Table 3 provides means of the pre-and post-test of SVT and MAT scores for the instruction subsample by grade and by sex. Table 4 includes the univariate P values resulted from the multivariate and univariate analysis of repeated measures. Two planned comparisons were used for the grade main effects--(7,8) vs. 6 and 7 vs. 8. Table 5 provides means of the post-and retention-test of SVT scores for the retention subsample by grade and by sex.

Table 3

Pre-and Post-test means of SVT and MAT scores for the Instruction subsample by grade and by sex

		SVT		MAT	
		Pretest	Posttest	Pretest	Posttest
	N	M	M	M	M
Grade 6	108	11.04	16.96	3.094	3.153
Boys	54	12.48	18.83	3.156	3.265
Girls	54	9.59	15.09	3.032	3.040
Grade 7	142	11.22	20.02	2.918	2.947
Boys	74	11.74	21.00	2.761	2.815
Girls	68	10.64	18.96	3.089	3.091
Grade 8	180	13.23	20.56	2.779	2.737
Boys	79	15.98	23.06	2.816	2.753
Girls	101	12.11	18.60	2.742	2.725
Total	430	12.01	19.48	2.904	2.911
Boys	207	13.48	21.12	2.889	2.909
Girls	223	10.65	17.86	2.918	2.913

Table 4

Analysis of Repeated Measures--P Values

	Grade by Sex by Time	Sex by Time	Grade by Time		Time effect
			(7,8)vs.6	7vs.8	
	P<	P <	P<	P<	P <
SVT	.5388	.2927	.0002	.0057	.0001
MAT	.5508	.7415	.3160	.3150	.8195

It was concluded from the analysis of the pre-post data that after the instruction: (1) sixth, seventh, and eighth grade boys and girls performed significantly higher on the spatial visualization test; however, no change in attitudes toward mathematics occured. (2) boys and girls gained similarly from the instruction, in spite of initial sex differences. (3) seventh grade students, regardless of sex, gained more from the instruction than sixth and eighth graders. In addition, the retention of the effects of the instruction persisted; after a four-week period, boys and girls performed higher on the spatial visualization test than on the posttest.

Table 5

Post-and Retention-test Means of SVT Scores for the Retention Subsample by Grade and by Sex

		Posttest	Retention
	N	M	M
Grade 6	52	17.11	18.65
Boys	26	18.54	19.50
Girls	26	15.70	17.81
Grade 7	79	20.06	20.71
Boys	41	20.93	21.46
Girls	38	19.13	19.89
Grade 8	107	21.23	22.36
Boys	45	24.40	25.02
Girls	62	18.94	20.42
Total	238	19.95	21.00
Boys	112	21.77	22.44
Girls	126	18.33	19.72

FACTORS AND MULTIPLES

This unit is a sequence of eight activities built upon the fundamental theorem of arithmetic and its related concepts. Through a series of games and challenges the students investigate primes, composites, factors, and products. The intention is to provide a basis for what Skemp would call a relational understanding of these ideas which will serve the students throughout their mathematical lives. The activities include a factor game, a product game, finding factor pairs, the sieve of Eratosthenes, unique prime factorization, common factors and multiples, and a game of paper pool which uses all of the ideas. Extensive use is made of calculators.

The evaluation of the unit was similar to that of the Spatial Visualization unit. Three different schools were used. Grades five, six and eight were taught in two Lansing schools with large low S.E.S. populations and grade seven was in a suburban middle school.

Table 6 provides means of the pre-and post-test of the Factors and Multiples Test (FMT, range 0-25) and the Mathematics Attitude Scale (MAT, range 1-5) by grade and by sex. It also includes the P values resulted from the univariate analysis of repeated measures. The results showed significant growth from pre to post test at all grade levels. There were no significant sex differences.

Table 6

Pretest and Posttest Means on FMT and MAT and P-Values from the Univariate Analyses of Repeated Measures

		FMT		MAT		TIME EFFECT		GRADE	
		Pre	Post	Pre	Post	FMT	MAT	FMT	MAT
	N	M	M	M	M	P<	P<	P<	P<
School 1									
Grade 5	102	6.90	11.04	3.578	3.510	.0001	.0129	.0060	.1143
Boys	51	6.57	11.45	3.330	3.347				
Girls	51	7.24	10.63	3.827	3.674				
Grade 6	91	10.42	12.78	3.266	2.984				
Boys	47	10.43	11.83	3.181	2.848				
Girls	44	10.41	13.80	3.356	3.129				
School 2									
Grade 7	101	12.04	14.51	2.688	2.816	.001	.0802		
Boys	43	12.65	15.12	2.595	2.845				
Girls	58	11.59	14.07	2.756	2.794				
School 3									
Grade 8	148	10.32	11.95	2.718	2.846	.0001	.0687		
Boys	81	9.57	10.83	2.572	2.701				
Girls	67	11.24	13.31	2.916	3.021				

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ESTIMATION AND APPROXIMATION - RESULTS AND RECOMMENDATIONS FROM RECENT RESEARCH

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This paper describes a recent research study which examined literature relating to estimation, approximation and the use of the calculator in schools, and the existing state of use of these concepts in a major subset of school boards in Ontario by means of a questionnaire and subsequent document and text analysis.

The material obtained was analysed using the five categories of: meaning, rationale, research, curriculum, and teaching strategies. A framework, together with some teaching suggestions was developed for an integrated approach to including this area of mathematics in the school curriculum, with a major recommendation that in-service education for teachers is provided.

The questionnaire responses and document analysis indicated that provision was being made for the topic in the Ontario curriculum, with significant reliance on commercial texts. Integration with the use of the calculator was only a reality in one or two boards, and in-service education was mainly by means of workshops, and apparently seldom on a regular basis.

PURPOSE AND SCOPE OF THE PAPER

This paper describes a recent research study the the author conducted throughout 1981 and 1982.

The main aims of the study were to establish more clearly what was meant by these concepts, find out to what extent they were used in Ontario, and study their relationship to the use of the hand held calculator. The study had two parts:
i) a fairly comprehensive survey of the literature relating to estimation, approximation, and the use of the hand held calculator as elements of the school mathematics curriculum,
ii) a study of the present state of affairs with respect to these topics in Ontario schools from Grade 4 through 10 by means of a questionnaire

and analysis of it and relevant curriculum documents and texts.

THE LITERATURE SURVEY

In all, summaries were compiled of materials from 6 N.C.T.M. Yearbooks, (1937 to 1982), 11 research studies (1966 to 1982), 9 journal articles (1976 to 1982), 10 books (1976 to 1982), and 3 compilations (one each of articles, references, and activities). Additionally, definitions and descriptions of the meanings of the terms (i) estimation and (ii) approximation were culled from two standard dictionaires and one scientific encyclopedia.

The abstracted summaries grouped themselves fairly naturally into one of the following five categories:

1. The Meaning of Estimation, Approximation and Related Concepts
2. Rationale for Studying These Concepts at School
3. Relevant Research
4. Suggestions for Curricular Experiences
5. Some Teaching Strategies

Estimating, according to the dictionary seems to be a loose term, but when applied to school mathematics, it seems to imply number or numerosity. Bright (1976) considers estimating 'as a mental process of arriving at a measurement or measure without the aid of measuring tools.' Siegel (1982) broadens the definition to include counting as well as measurement while Atweh (1982) regards it as one aspect of mental arithmetic.

Approximation, has a more restricted meaning: "being close to" as opposed to "roughly", and is generally associated exclusively with the realm of number. Payne and Seber (1959) are more concerned to allow understanding of these concepts to grow via a set of experiences.

When one considers the ideas of "checking" and "reasonableness of results", there seems to be need for clearer understanding of whether the reasonableness is being determined before or after the calculation. Johnson (1979) distinguishes between "real-world" and "mathematical" reasonableness.

Finally, with the emergence of the hand held calculator, we need to be clear about when we are estimating, checking or approximating, and how this affects the intelligent use of the calculator.

A strong rationale for including teaching of these concepts has been provided by Skvarcius (1973), Bright (1976), Siegel (1982) and others. In particular, estimation facilitates reasonableness of results; is an integral part of the division algorithm; is related strongly to mental arithmetic; is an indispensable part of the everyday life of both the individual adult and society in planning; and is essential for skilled measurement and for solving problems successfully.

Research seems directed either at better understanding of the nature of estimation or at evaluating the effects of instruction in it on computational and other competencies. Schoen (1981) summarised the results of a literature survey by him in the following way:

> "The variables that appear to be related to estimation skill are general intelligence and mathematical ability. There is evidence that estimation skill is less strongly related to computational skill and verbal problem-solving ability, and there appears to be little relationship between the ability to estimate lengths or areas and the ability to do computations rapidly."

Reys and others (1982) defined 'good computational estimators' by means of a test, and identified three processes (reformulation, translation and compensation) used by them together with other characteristics such as tolerance for error, knowledge of arithmetic properties and self-confidence.

Siegel (1982) tested out a competence model of estimation based on rational task analysis which identifies (i) "benchmark" estimation and (ii) "decomposition/recomposition" estimation. He claims that (ii) occurs only if (i) yields no suitable reference point. Siegel found a need to revise the model to make it hierarchical.

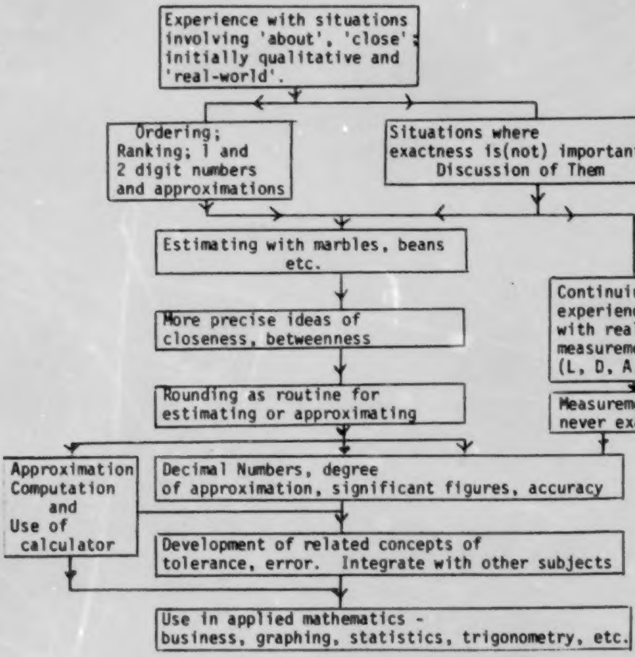
The results of a number of studies on the effect of instruction indicate that it is effective in improving skill in estimation but most employed short periods of instruction, and no firm conclusions emerge. Studies involving the use of the calculator so far seem to suggest that it is of positive value in its impact on learning to estimate, but the evidence is sketchy and many more studies are needed.

The disappointing N.A.E.P. results and other evidence indicating that students do not see the need to estimate suggest that the concept of estimation should be carefully developed in the early years of schooling, with continuing attention being paid to the relationship between estimating and using the hand held calculator. Also, teachers must be able to distinguish each concept and its particular function in learning mathematics.

A rough sequence of curricular experiences, in line with Crawford (1982), it outlined in Figure 1.

FIGURE 1

ROUGH OUTLINE OF SUGGESTED CURRICULAR EXPERIENCES


Experience with situations involving 'about', 'close'; initially qualitative and 'real-world'.
Ordering; Ranking; 1 and 2 digit numbers and approximations
Situations where exactness is(not) important; Discussion of Them
Estimating with marbles, beans etc.
Continuing experience with real-world measurement (L, D, A, V, etc.)
More precise ideas of closeness, betweenness
Rounding as routine for estimating or approximating
Measurement is never exact
Approximation Computation and Use of calculator
Decimal Numbers, degree of approximation, significant figures, accuracy
Development of related concepts of tolerance, error. Integrate with other subjects
Use in applied mathematics - business, graphing, statistics, trigonometry, etc.

Note: Use calculator in appropriate places.

Good teaching strategies are outlined in several of the books, yearbooks, and journal articles surveyed, mostly related to estimation - activities (Skemp 1977); Calculator Club, (Drake 1978), Problem-Solving (Morris 1981). As regards approximation, an excellent sequence of learning the concept of approximation is developed in McHale and Witzke (1977) and background material abounds in the various yearbooks, especially those of 1959 and 1976. Reys and Bestgen (1981) also offer excellent suggestions.

QUESTIONNAIRE SURVEY AND DOCUMENT ANALYSIS

A four part questionnaire dealing with estimation, approximation and the use of the hand calculator was developed in February 1981, and completed by 9 of 16 Ontario School Boards employing full-time Mathematics Coordinators. This response represented 60% of the students in these boards, and about 25% of all Grade 4 through 10 students in Ontario. The Boards surveyed are generally regarded as the most wealthy, influential and progressive Boards in the province. Tables were compiled summarising the responses relating tocurriculum development and implementation; use of texts; estimation and approximation in the curriculum, use of the hand-held calculator; and future plans.

Most boards initiate curriculum development and take several years to implement a given program. Difficulties in implementation include lack of time and money, and vairable teacher backgrounds. Workshops form the main method, but the extent and depth of on-going systematic in-service education appears low.

Estimation and approximation is formally included in the Grades 7 to 10 curriculum and all responding Boards use commercial texts significantly, with some use of "home-grown" materials. Analysis of four series of texts reported as used by six or more of the 9 responding boards revealed a reasonable and intelligent coverage of the topics, mainly in a computational as opposed to problem-solving setting. One of the two series for Grades 7 through 10 dealt very capably with reasonableness of answers in problem-solving, selecting suitable units of measure, estimating and real life, and rounding and checking.

Policy with respect to the use of the hand held calculators was highly variable, and integration of its use into the curriculum is still the exception rather than the rule, although two boards have developed their own guidelines incorporating its use very effectively (at least on paper).

Boards expect the use of the hand calculator to increase in the near future, but there is increasing emphasis on using microcomputers, and so there is some danger that the necessary integration between estimation and the use of the calculator may not receive continued attention.

SUMMING UP

The study revealed a considerable body of literature on the concepts of estimation and approximation in the school mathematics curriculum in North America. According to research, it appears that estimation is related to general intelligence, and mathematical and verbal ability, that it is somewhat task-specific and that it does lead to improved computation. Attempts continue to be made to define it more clearly as a process and/or skill. The identification of the estimation processes of reformulation, translation and compensation by Reys and Bestgen and the procedures of benchmark estimation and decomposition/recomposition found by Siegel and others give some good bases for constructing curricular and teaching sequences. This can be supplemented by the skill analysis of the I.T.E.M.S. program, and the work of people such as Payne and Seber and Bright, using the kind of curricular framework previously outlined by Crawford.

Good sources for curricular material including work with calculators exist (e.g., McHale and Witzke, Moursund, Morris). More needs to be done, however, to widen the applications of these concepts to areas of real life. In particular, the everyday use of estimates and approximation, the significance of them in science and the changes which high technology is making in levels of precision and accuracy all need continuing and intelligent research and attention.

All of these aspects need to be combined in a more concerted and systematic drive to educate teachers to the meanings and significance of the concepts of estimation and approximation and how they relate to the intelligent use of the calculator, and also of the computer. According to the survey of the situation in Ontario, the time is ripe for sustained effort of this kind.

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A CONSTRUCTIVIST MODEL OF UNDERSTANDING

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Initial models attempting to describe understanding have mainly focused on the use of rules and the solving of problems. As such they have proved to be inadequate to describe the understanding involved in concept formation. The model suggested here accounts for the constructive nature of concept formation and describes four levels of understanding corresponding to four stages in such constructions, namely an intuitive level, a procedural level, a level identified with abstraction and one with formalization.

Last year, at the meeting in Minnesota, we reported some problems which revealed themselves in our first experiment dealing with the training of teachers in the analysis of mathematical concepts (Bergeron & Herscovics,1981). In that experiment we used a model of understanding which was an adaptation of the tetrahedral model (Byers & Herscovics,1977) which itself was, to some extent, a synthesis of Bruner's (1960) and Skemp's (1976) previous work. Unfortunately, our hybrid model proved to be ill-suited for the analysis of concepts since the criteria chosen to describe modes of understanding dealt essentially with the use of rules (with or without justification) and the solving of problems (with or without awareness of the strategies used). In a second communication we sketched a tentative outline of a new model better suited to describe understanding within the context of concept formation (Herscovics & Bergeron,1981). During this last year's work, this new model has taken shape.

THE CONSTRUCTION OF MATHEMATICAL CONCEPTS

An examination of research on the acquisition of fundamental concepts taught at the elementary level shows that their construction stretches out over a period of many years (Piaget,1947; Ginsburg, 1977). This is why it is difficult to study this kind of learning within the context of the usual paradigm dealing with concept formation, which limits itself to defining a concept by a list of "critical attributes" (Tennyson & Ok-Choon Park,1980). Much of the research issued from this perspective concerned itself with finding the optimal number of examples and non-examples needed to identify a given concept. In some instances, this approach could prove to be of some value as with the notion of even and odd number. However, the acquisition of such a concept presumes that other more fundamental ones, such as number and divisibility, are already available to the learner. But this traditional approach can hardly inform us about the acquisition of number or divisibility since these are notions which, from a cognitive point of view, cannot be defined by a list of critical attributes. Indeed, these are very complex concepts involving a great many other ones which need to be related into a cognitive network. This is why we use the words "notion" and "concept" as synonymous with "conceptual schema", an expression which better translates this idea of the cognitive networks which can be associated with each fundamental concept in mathematics. It is rather in the theories of information processing (Simon,1979) and Piagetian constructivism (Piaget,1972) that one can trace some influence on our work. These psychological theories, as well as other considerations pertaining to some specific needs of mathematical didactics, have led to the identification of four levels of understanding: the first one, that of intuition, a second one involving procedures, the third dealing with abstraction, and a last level, that of formalization.

THE INTUITIVE LEVEL

For most of the notions taught, one can find some preconcepts which can be viewed as embryonic to the conceptual schema whose construction is intended. For instance, the bringing together of two sets of objects or the action of adding to a given set are preconcepts of the notion of addition. There is not yet any quantification, maybe at most some simple estimation. These are situations which lead to what Ginsburg (1977) describes as "informal mathematics". Even without any schooling, through such experiences children can become aware that "the whole is always larger than any of its parts". Of course, they cannot as yet verbalize such logico-mathematical principles and it is rather through their actions that they express their thinking (Ginsburg & Opper,1979). And at this level, as has been observed by Piaget, their thought is essentially based on visual perception (Piaget & Szeminska,1967; Piaget & Inhelder,1977). In summary, the level of cognition which we call "intuitive" is evidenced by informal knowledge which is characterized by preconcepts, a type of thinking based on visual perception, and primitive unquantified actions limited to rough approximations.

Several pedagogical and psychological reasons justify the selection of this informal mathematics as a first level. From a pedagogical point of view this stand induces a search of appropriate situations from the child's experience for each intended concept. By using these situations as a starting point, we provide the acquisition of new knowledge with meaning and relevance. This can have some psychological repercussions such as increased motivation. From a cognitive point of view, we thus make sure that the child's mathematics are solidly rooted in his world and experience. Moreover, it allows him to proceed with the construction of his conceptual schemata without any cognitive discontinuity.

THE ACQUISITION OF MATHEMATICAL PROCEDURES

Of course, this informal mathematics cannot suffice for it allows at most some rough approximations. In fact, it can only be used to initiate a following stage of mathematization which will fit together and coordinate the intuitive knowledge and some prerequisites in an initial procedure which will render this knowledge more operational and more precise. The assimilation of such a procedure constitutes a second level of cognition which we call "procedural understanding". For example, in the case of addition, the most elementary procedure consists in bringing together two sets of objects and counting them starting from one. While intuitive knowledge may result from situations the child could have experienced outside of school, it is obvious that procedural understanding results essentially from pedagogical interventions. Briefly, the level of procedural understanding is evidenced by the acquisition of a procedure which, by coordinating intuitive knowledge and some prerequisites, constitutes a first formal construction of a given concept.

With this second level of cognition, that of procedural understanding, we in fact accept a philosophy which is implicit in information processing. In accordance with this viewpoint, we believe that a concept cannot be formally constructed without a certain procedure. However, our approaches differ since rather than being interested in the simulation of such procedures, we consider these in a larger framework by viewing them as only one stage in the process of constructing conceptual schemata.

ABSTRACTION

The type of understanding which we associate with a second level of cognition deals essentially with the acquisition of a procedure. Such procedures bring about the formal construction of mathematical notions by which we mean mathematical objects such as number or mathematical transformations such as the operations of arithmetic. These are but initial procedures and at the very beginning the intended concept is blurred and confused with the procedure leading to its construction (for instance, at the beginning, the notion of number is confused with the counting procedure). It is only very gradually that the "outline" of a concept gains precision, that it separates from the procedure, and that it starts having an existence of its own in our mind. This describes a process of abstraction which we identify as a third level of understanding.

The particular nature of mathematics involves two kinds of abstraction that Piaget (1973,p.81-82) calls "empirical abstraction"(dealing with the physical properties of objects), and "reflective abstraction"(dealing with actions and their coordination). For example, the child who does not conserve number, even if he can count, has but extracted a pseudo-physical property of the set for he still is influenced by the physical arrangement of the objects. On the other hand, the child who is aware that number is invariant with regard to the configuration or the order in which the objects are counted gives evidence of reflective abstraction. Of course, the construction of such invariants necessitates the coordination of several variables as well as the reversibility of thought. It is precisely such a process that characterizes mathematical activity which explains why Piaget considers it to be synonymous with "logico-mathematical" abstraction. This is our third level of understanding and, to sum up, it manifests itself either by the generalization of a concept, or by conservation which reflects its invariance, or by the reversibility of the transformations involved.

FORMALIZATION

The fourth level of understanding, that of formalization, takes into account the particularly symbolic nature of mathematics. Although "formalization" is often interpreted in the sense of axiomatization or formal proof, we also attach to it a third meaning. In this word we can retrieve the idea of form which can be distinguished from content. This is an important distinction for by "content" we refer to "mathematical notions" and by "form" we refer to the "representations given to these notions".

Several recent studies have shown that the symbolic representation of mathematics creates specific cognitive problems, and this at all levels from the primary school right up to the university. For instance, Ginsburg (1977) and Carpenter & Moser (1979) have reported that many children could handle word problems as long as they did not have to deal with them symbolically. Herscovics (1979) has indicated similar results for algebra at the secondary level and so did Clement (1982) with university students. Since notation brings about an increase in cognitive problems,one might be tempted to identify symbolization as a fourth level of understanding. But the work of Erlwanger (1973,1975) on individualized programmed instruction has shown the extent to which students could succeed on some tests by manipulating symbols which were devoid of any meaning, basing themselves solely on the disposition of the symbols to derive idiosyncratic rules. We thus must conclude that, by itself, the correct manipulation of symbols cannot be taken as a criterion of understanding. This has led us to consider symbolization as relevant to a fourth level of understanding only if prior abstraction of the concept has occurred. Thus, depending on the notion intended, we consider that formalization is evidenced either by the use of symbolism, or the logical justification of operations, or by the discovery of axioms. But in all cases, this level presumes prior abstraction.

CONCLUSIONS

The identification of different levels of understanding has many valuable pedagogical implications since it brings about an awareness of the many steps involved in the construction of a conceptual schema. For instance, how could one hope to teach multiplication in terms of repeated addition if a certain prior abstraction of addition is lacking?

The criteria we have used to describe our model could be regarded as cognitive objectives in the teaching of mathematics. Of course, this raises new questions regarding the kind of pedagogical interventions likely to bring about a progression from one level to the next. We can already foresee that the most difficult problem will be to determine the type of activities which favor the occurrence of reflective abstraction.

Although this model suggests many pedagogical questions, its constructive nature may lead to some incorrect conclusions with respect to teaching. For instance, one might gather that since formalization is considered as a last level of understanding, one should delay the introduction of symbolization. While we attach a great importance to this form of representation in view of its prime function in the mathematical development of the pupil, we would agree that a premature introduction of symbolization can be counter-productive. Indeed, what would be the sense in teaching numerals to a child who cannot as yet count ? However there is no harm in introducing them after counting has been mastered. But it is obvious that according to whether the student conserves number or not, the meaning of a numeral will be different. Another possible misconception would be to consider the model as linear in the sense that all four levels of understanding of a concept have to be attained prior to tackling the next concept. Of course this is not so, since much arithmetic is within the reach of the child well before he has become a conserver.

The various levels of cognition described in this model have been useful in our attempt to integrate in an overall synthesis the results of prior research on addition (Herscovics & Bergeron,1982), and on the concept of function (Bergeron & Herscovics,1982). From a more practical point of view, we have used it in training teachers to analyse certain concepts they teach in elementary school.

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RELATIONAL AND INSTRUMENTAL UNDERSTANDING OF MATHEMATICS: AN ASSESSMENT PROJECT

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Following a study by Richard Skemp in Great Britain, a 20 item, group administered test was developed (with two parts to each item) to assess the type of mathematical understanding (relational or instrumental) possessed by a group of secondary school students. The test was administered to 221 ninth and tenth grade students. Comparison of test responses with interview data from a random sample of the students suggest that the first part of items may be assessing students' relational or instrumental understanding in the reflective mode, whereas the second part of items may be assessing students' intuitive level of mathematical understanding.

Relational and instrumental understanding of mathematical concepts have been distinguished as different types of understanding by several math educators and psychologists. Relational understanding can be thought of as "knowing both what to do and why," whereas instrumental understanding is simply "knowing how." The former has been linked with successful learning of mathematics in terms of retention, adaptability and transfer. The latter has been linked with short-term memorization of facts. Richard Skemp (1979) has suggested that associated with each type of understanding are two different modes of mental activity: the intuitive and reflective modes.

The National Foundation for Educational Research in Great Britain began a feasibility study for the design of a written, group administered test to assess the different types of understanding. This project is an elaboration of that NFER study.

A modified test was developed containing 20 items covering a wide range of secondary school topics (see Table 1). The first eight items on the revised test (TUM) were revisions of items used in the NFER study.

TABLE 1
Domain Specifications for the Test for
Understanding Mathematics (TUM) and Corresponding Items

Mathematical Domains	Item Labels[a]
Addition of fractions	A
Decimals (rounding)	B
Division of whole number by a rational	C
Place value (multiplication by the base number)	D
Area measure	E
Addition of negative integers	F
Decimals (fraction equivalence)	G
Quadratic equations	H
Linear equations	J
Linear measure	K
Estimation - reasonableness of results	L
Division of rational numbers	M
Distributive property of arithmetic	R
Ratio and proportion	S
Ratio and proportion	T
Multiplicative Identity	V
Multiplicative Inverse	W
Topology (open/closed curves)	X
Angle measure	Y
Logic and sets	Z

[a] Items A, B, C, D, E, F, G, H are the eight items from the NFER study.

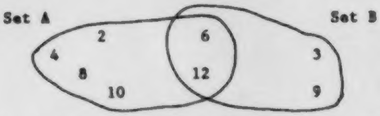
Each item consisted of two parts. Part one asked the student to choose one of three explanations, reflecting either relational, instrumental or non-understanding of a stem statement (see Figure 1).

Item Z Part One

Stem Statement: A = {2,4,6,8,10,12}
B = {3,6,9,12}
so A∩B is a set with two members.
Why?

Explanations: (Check the one which makes most sense to you.)

___ 1.



6 and 12 are in both set A and set B. No other numbers are in both sets, so the intersection of set A with set B (A∩B) has only 2 members: 6 and 12.

___ 2. Set A has 6 members, set B has 4 members. 6 - 4 = 2 so A∩B has 2 members.

___ 3. A∩B is the set of all elements common to both set A and set B.

___ 4. None of them make sense to me.

Figure 1

Part one of a sample item

Part two asked the student to choose one of three mathematical problems which had either a deep mathematical connection with the same stem statement or had only a superficial (symbolic) connection with the stem statement. The student was not asked to solve any of the problems, but to choose the one which was concerned with the same kind of mathematical thinking as the stem statement (see Figure 2).

Item 7 Part Two

Stem Statement: $A = \{2,4,6,8,10,12\}$
$B = \{3,6,9,12\}$
so $A \cap B$ is a set with two members.
Why?

Which of the following is basically concerned with the same mathematical thinking as the stem statement?

___ a) 2, 4, 8, 10, 14, 16, 18, 20.
Find the missing numbers in this sequence.

___ b) All two footed animals are called bipeds.
All mammals are warm blooded.
Is a giraffe a warm blooded biped?

___ c) Find all the common factors of 720 and 105.

Figure 2

Part two of a sample item

METHODOLOGY

Subtests of ten items were administered, during a pilot study, to 97 ninth and tenth grade students. An interview protocol was developed and tested on 13 of these students. The complete test was administered to the main study group of 124 ninth grade students from a large metropolitan area.

A random sample of 41 students from the main study group was interviewed during a four week period, commencing three weeks after the test administration. In the interviews, students were asked for their own explanations of the stem statements on each item. They were also asked again to choose one of the three explanations reflecting either relational, instrumental or non-understanding of the stem statement for that item, and then to justify their choice.

The interview responses were used as criteria for answering the following questions concerning the written test:

1. Can the deep/superficial choices on part two of the test items be used to predict relational and instrumental understanding of students?
2. Can the relational/instrumental/non-understanding choices on part one of the test items be used to determine those students who have little or no understanding of the mathematical concept?
3. How close a match is there between test and interview for each student across items, for each item across students and for all students across all items?

A match analysis was conducted which tested the obtained number of matches between test and interview against the number expected by chance alone. Items which generated a match between interview and test for more than half the students were selected for a chi-square analysis summed across items and students. Comparisons were made between responses to the different parts of the written test, different modes of the interview, and between pilot and main study groups.

RESULTS

Eight of the 20 items selected by the match analysis generated a level of prediction (69%) similar to that of the NFER study, for the prediction of relational and instrumental understanding from the deep/superficial choices on part two of these items.

Fifteen of the 20 items selected by the match analysis demonstrated a strong association between choices on part one of the test items and justification responses in the interview (chi-square > 150). This result is different from that obtained in the NFER study (low association between part one and the interview). Non-understanding choices on part one of the test items were unreliable indicators of non-understanding as determined by the interview.

Students in the main study group chose mostly instrumental explanations on part one of items but selected more deep (than superficial) options on part two of items. The tenth grade students in the pilot group chose mostly relational explanations on part one and deep options on part two of items. The differences in response types between the two groups were significant.

DISCUSSION

The results suggest that ninth grade students' responses to part one of the test items may be indicative of explicit instrumental understanding, whereas their responses to part two of the test items may be indicative of an intuitive level of relational understanding.

Recommendations are made for the modification of part one of the test items in order to determine more accurately those students who lack understanding at any level. Further studies are suggested to investigate the use of the modified part one of test items for determining students' explicit (or reflective) understanding, and to investigate the use of part two of the test items for predicting students' intuitive level of understanding. Revisions are also recommended for some of the part two options.

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QUESTIONS ON SOME ISSUES IN THE PSYCHOLOGY OF MATHEMATICS INSTRUCTION

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INTRODUCTION

In his invited talk at the AERA meeting at New York City, March 1982, Robert Gagné gave rise to a renewed discussion of the cognitive phenomena involved in children's learning of mathematics, mainly focussed on arithmetic, and of implications for instruction. According to Gagné, attention has to be given to each of the three phases of a performance model as follows: (1) Translation of a verbally described situation into mathematical form; (2) Central computation phase; (3) Validation of the solution.

In particular he argued toward the maxim of an automaticity of skills which are involved in the computation phase, in order to bring about optimal performance. He ultimately stated that the cognitive capability required for students from kindergarten through highschool is a set of stored rules which are best carried out automatically, and that lots of practice should be devoted toward automatizing such rules (Gagné, Note 1).

While the decade of the 1970's was characterized by the "back to the basics" movement, and the product-versus-process question is still being debated, these statements of Gagné might prove harmful in that they are likely to cause misunderstandings, give wrong impressions to teachers, and cause a precipitate repudiation of Gagné's ideas by math educators.

The purpose of this paper is to give a more careful inquiry into Gagné's proposals, and at the same time take up the discussion of skill automaticity in a new light. On the one hand, this is done to mediate the controversy between the views of mathematics educators--best expressed in the terms "learning with understanding--and the seemingly unpopular theses of Gagné. On the other hand, this discussion shall raise Questions on Some Issues in the Psychology of Mathematics Instruction which, in the author's opinion, require further research attention.

WHAT IS A SKILL?

"A skill is what a learner should be able to *do*." This pragmatic answer is quoted from Suydam & Dessart (1980), where an extensive discussion of skill learning in mathematics is given. They continue, "Skills arise from concepts and principles and provide a foundation for the development of other concepts and principles. Conceptual thought is derived in part from the understanding attained as skills are developed." (p. 207) This view shows intertwined dependencies between skills and conceptual knowledge. In fact, "The relationship between computational skill and mathematical understanding is one of the oldest concerns in the psychology of mathematics." With these words Resnick and Ford (1981) open their outlining remarks on the concerns of a psychology of mathematics instruction, and they continue, "Instead of focusing on the *interaction* between computation and understanding, between practice and insight, psychologists and mathematics educators have been busy trying to demonstrate the superiority of one over the other. [...] The *relationships* between skill and understanding were never effectively elucidated." (p. 246; emphasis theirs). Certainly, an attempt to analyze this relationship should also include mathematical skills other than computation.

Modern cognitive psychology as for example represented in the work of Anderson (1980), tends to view knowledge categorized as *declarative knowledge* and *procedural knowledge*. Declarative knowledge comprises the facts that one knows; and procedural knowledge comprises the skills one knows how to perform (p. 22 ff). The declarative-procedural distinction graphically supports the fact that two different things are of concern in any learning process. Declarative knowledge is what is stated in propositions and can be verbally communicated, for example, $3 + 5 = 8$, or $(a + b)^2 = a^2 + 2ab + b^2$, or "To solve quadratic equations, one method is to complete the square." This type of knowledge is viewed to be stored long-term in what is called semantic or factual memory. Skills, as procedural knowledge, are also stored in long-term memory ("procedural memory"), however, rather as an ordered sequence of actions that are performed when recalled. The term cognitive skill, (as opposed to motor skill) refers to the ability to perform various intellectual procedures. For example, writing out the steps necessary to solve a quadratic equation by completing the square is a cognitive skill.

The learning-with-understanding approach of today's mathematics education has led to the view that a learner shall be put in the position to do mathematics--for instance, solve a problem--by deriving it meaningfully from declarative knowledge. However, it is the procedural, not the declarative, knowledge that governs skilled performance.

AUTOMATICITY OF SKILLS

Taking his three phase model as a basis to achieve good problem solving, Gagné (Note 1, Hypothesis 2) claims that "performance is likely to be best if the phase of computation is done automatically". In illustrating this, he makes reference to the assumption of a working memory existing in the human mind, in which one is to store those items of cognitive processing where attention is directed to, and that this working memory is limited (further reference is found in Resnick & Ford, 1981). Hence, "the scarce cognitive resource of attention needs to be devoted to the most intricate and complex parts of the task, if its solution is going to be efficiently and successfully arrived at." (Note 1, p. 17). Gagné then remarks that automatization of some kinds of algebraic operations gives rise to rapid and fluent thinking in applicational situations, and that differences in speed and accuracy of computation found among students is due to a lack of automaticity of computational skills in some of them. In a problem solving situation a higher amount of attention would then have to be devoted to skill performance which would interfere with and slow down the cognitive operations devoted to solving the problem.

Anderson (op. cit. p. 226) lists three steps in which skill learning occurs: (1) The cognitive stage (declarative), in which a description of the procedure is learned with the aim to understand it; (2) The associative stage, in which the declarative information is transformed into a procedural form, that is, a method for performing the skill is worked out; (3) The autonomous stage, in which the skill becomes more and more rapid and automatic (by way of practice); verbal mediation in the performance of the skill often disappears at this point. It may be questioned -- and this is Gagné's concern--whether trends in curriculum development tend to ignore the importance of the third stage. Emphasis is put on the acquisition of factual (declarative) knowledge. Understanding of a task is said to enable the learner then to "figure it out". Time-and cost-effectiveness standards are applied to the use of instructional time, and "too much" practice is avoided; see, for example, the "Agenda" of the National Council of Teachers of Mathematics (1980, pp. 6-12). Further practice of a skill is left to problem-solving situations that require application of that skill. However, this is a case where too much spacing can be counterproductive.

Stage 3 of the above listing comprises the promotion of a skill to an automatized skill which is autonomous in the sense that it does not even require a verbal correspondant in declarative form. Not only does efficiency of a skill increase as it gets more and more automatic, but also mental effort in performing the skill decreases; there is less need to monitor each action with care. Suydam and Dessart (op cit.) refer to the fact that mastered skills require relatively little reflection as they become "reflexes" which are conducted at a subconscious level. As the demands on conscious resource diminish, a reduction in mental work load is achieved.

It was demonstrated (cf. Anderson, op. cit.) that highly practiced skills cease to interfere with other ongoing behaviours; that is, different cognitive activities can be conducted simultaneously. Anderson moreover indicates that automatizing subskills before going on to higher skills contributes to learning higher skills (in computer programming; cf. p. 249 ff). The reason apparently lies in the relationship between attention and automaticity: As subcomponents in programming become more automatic, programmers can focus more attention on higher level problems.

ROTE SKILLS OR MEANINGFUL SKILLS?

Given any skill, what makes the difference between its being (a) a rote skill or (b) a meaningful one? We are to distinguish two cases, in our analysis of each. First, the skill in question is not automatized; second, it is automatized.

1a) A rote skill that has not been automatized requires the recall of a rule, or a set of rules, which is then applied mechanically, without understanding. If part of the rules, or knowledge of how to instantiate the variables, is forgotten, there is no way to reconstruct it from other knowledge. (For instance, under these circumstances this often happens with the quadratic formula.)

1b) A skill derived meaningfully and that has not been automatized makes reference to factual knowledge that a learner has, in order to apply this for a certain purpose. (For instance, a learner knowing how to represent numbers with Dienes blocks can use this to derive the subtraction algorithm, including "borrowing".) If such a skill has not been practiced to the point of recall, it can be viewed as a problem solving process (cf. Wachsmuth, 1982). Instead of dealing with symbols directly (or "concretely", as Gagné calls it), it is dealt with symbols indirectly (or "abstractly", as Gagné calls it); namely, it is actually dealt with the referents of symbols.

2a and 2b) The idea of practicing a skill to the point of recall ("over-practicing") is the following: The *sequence of actions* necessary to do something is made to occur automatically; without having to derive, or reconstruct, any action from an understanding or recall of facts, purposes, and goals. In this process, one step triggers the next; and it is just the ability to perform this sequence of actions in order to achieve a certain thing which makes up the skill.

The point was raised earlier that children's maturity to fully understand why and how a skill works may appear at a later time than they are able to perform rote skills; e.g., see the discussion on fraction algorithms of Suydam (1979, pp.302/3). So the question is whether it could be of advantage if automatized rote skills precede a full understanding of such skills.

When an automatic skill is called upon, action is not derived from understanding how the procedure works. It is rather guided by its own action scheme. To be very clear: This is like the difference between understanding how bike-riding works and being able to ride a bike. The common fact that practicing of motor skills should not be interfered with by thinking about it seems to have a correspondant in the acquisition of procedural mathematic knowledge. Gagné's Hypothesis 1 (Note 1, p. 16) about the teaching of computational skills (omitting reinforcement of incorrect performance plus teaching correct rules again) is further supported by findings of E.G. Davis (1978): During drill sessions, it is best to emphasize remembering; don't explain. Understanding seems to be unnecessary for the *execution* of a procedure.

It is still controversial whether better understanding of a procedure gives rise to better performance; for the case of fraction algorithms, cf. Suydam (1978, p.302). Understanding, however, seems to be important for embedding a procedure in the context of a problem-solving task correctly. It might well make sense to teach skills both rotely and meaningfully, maybe in a spiral fashion. In summary:

1. Do we have to think about procedural and declarative knowledge as two related, however separate issues of mathematical instruction?
2. Do there exist optimal mental ages for automatizing skills in children, and for teaching children to fully understand what they do?
3. If the optimal age for automatizing skills is earlier than the optimal age for understanding a skill, how do we proceed in instruction?

VALIDATING THE SOLUTION

Under this heading, Gagné (Note 1) makes reference to the fact that students may trust in a result obtained by means of an algorithm, without inspecting the solution meaningfully which could reveal a faulty computation. The important topic adressed here concerns the different levels at which a person can deal with mathematical symbols. First, the syntactic level, at which one is to manipulate the symbols (here numerals) directly-- as "concrete" objects of thought -- according to certain rules, and totally removed from their meanings (as numbers). This is the normal situation when an algorithm is employed routinely. Second, the semantic level -- which Gagné calls "abstract" --, at which one deals with the symbols by referring back to their meanings. For example, in making use of the fact that 19 × 20 equals 19 × 2 × 10 one needs to understand how numbers are denoted by means of numerals, in the place value system.

What is really addressed with the topic of validating is the interaction between these two levels, which must be a well-functioning one if mathematics is to be employed meaningfully. It is the question of "The Influence of Semantic Content on Algorithmic Behavior", a topic investigated by Davis and McKnight (1980). The study reports a specific phenomenon encountered with third and fourth graders who, in general, could use the standard subtraction algorithm correctly, but were almost certain to make the error of "skipping over intermediate zeroes" in certain subtraction problems.

These children could clearly use Dienes blocks to represent numbers and explain trading, upon request. But this latent knowledge did not alert them in the algorithmic situation. The two levels at which one can deal with this problem -- syntactical and semantical -- appeared to coexist unconnectedly in the minds of the subjects.

A major question raised by the Davis/McKnight phenomenon is whether this is "essentially inevitable for most children of this age, or whether it is a consequence of an excessively 'algorithmic' (and meaningless) program of instruction" (p. 75). Gagné's position about ways of verifying solutions is that "it would be desirable if they [such ways] were deliberately taught" (Note 1, p. 12). But the actual problem is:

4. How can children be taught ways of verifying solutions, in the sense that their algorithmic behavior in situations that are governed by syntactical rules is monitored by their semantic knowledge?

5. Can such an interaction of syntactical and semantical levels of information processing be taught at all to children of certain grade levels?

As one of the ten areas of basic mathematical skills listed in the NSCM position papers on Basic Mathematical Skills (Note 2), the "Alertness to the Reasonableness of Results" is viewed an essential skill. Apparantly, alertness comprises the *automatic generation of warnings* or critics in the course of a task performance. So, if a curriculum developing both understanding and skilled performance is the objective, research efforts will have to be devoted to the question:

6. How can warnings be automatized so as to achieve a reliable, alert interplay between skill and understanding in mathematics?

Apparantly, Gagné's view that the cognitive capability called upon for the students in grades K-12 be a set of stored rules which are best carried out automatically is only part of the story. It is also required that they can be carried out meaningfully, and that alertness skills permit to "switch to the meaningful mode" when necessary.

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THE ABSTRACTION OF COUNTING UNITS

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The concept of unit, though abstracted from experiences of unitary things, does not itself contain sensory components. It has to be abstracted from the experiencer's own attentional activity. The mature concept of "ten" (and "hundred") is more complex: as collection it must be abstracted by reflection upon counting activities, as counting unit by reflection upon constituted collections. These differences are here explicated in terms of Piaget's theory of abstraction.

It seems to be a general observational fact that children acquire number words some time before they could be said to have any conception of what a number is. They learn some number words singly, as names of specific patterns, such as the numeral "3" or the famous "domino-five"; or as referring to specific situations, as when Mummy says: "Bring me two apples", which means coming back with an apple in each hand.

Parents, as a rule, take pride in their children's "counting", they teach it, and they reinforce it. Toddlers, therefore, learn to recite number words, one after the other, in a conventional order, like a prayer or a nursery rime. Many of the words that are learned in that way have no meaning for the children. Years may go by, for instance, before a child realizes that "our daily bread" has nothing to do with dairy farms. With number words it is much the same, but there is also a difference. The meaning of number words is actually no more abstract than the meaning of, say, articles or prepositions; but the use of number words in mathematics requires that their abstract meaning be consciously grasped, whereas with articles or prepositions it is sufficient if one learns how to use them. (Even professional linguists have rarely felt a need to investigate the conceptual meaning of these items.)

To become aware of the specific character of an abstraction requires an act of reflection in the special philosopher's sense of that term. "Reflection" was introduced by Locke to indicate a source of ideas "every man has wholly in himself" and, though he likened it to an "internal sense", he wanted it clearly distinguished from sensation:

> I call this REFLECTION, the ideas it affords being such only as the mind gets by reflecting on its own operations within itself. (1690, Book II, Ch.I, §4)

To graps the meaning of number words and other similarly abstract signs or symbols, the child that is acquiring the language of the people among whom he or she is growing up, must do three things:

1. Learn to recognize and vocally to produce the specific words.
2. Isolate, through verbal and other interactions with speakers of the language, specific situations in which these words are used.
3. Isolate specific perceptual or conceptual operations which one must carry out oneself in those situations in order to interact satisfactorily with others.

Each one of these three tasks requires a different type of _abstraction_, but only the last demands _reflection_ in Locke's sense, i.e. the active subject's awareness of its own operations. Piaget long ago characterized different types of abstraction, but only recently has he published a work that deals exclusively with this problem. The two volumes of _Research on Reflective Abstraction_ (1977) contain twenty empirical studies that exemplify his classification:

A. Empirical abstraction

B. Reflective abstraction
   1. Pseudo-empirical abstraction
   2. Reflective (but unconscious) abstraction
   3. Reflected abstraction or Reflective Thought

An example may illustrate this development. Before anyone can acquire a concept of number, there must be a concept of unit, because whatever else number words might be said to refer to, they refer to collections of units. "Unit", of course, is already an abstraction and, in the experience of children, it will usually be abstracted from things, e.g., experiential things, such as apples. Now, in order to experience an apple it is necessary to isolate a number of sensory signals from the rest of one's experiential field which, except during sleep or unconsciousness, is always teeming with sensory signals of every kind. Thus, there is a twofold requirement: specific sensory signals must be found present and attended to; and once they are attended to, they must be separated or cut out as a separate group from the experiential field. The one requirement concerns the characteristics of apple – appleness, if you will; the other concerns the unity of the experience, its thinghood (von Glasersfeld, 1981). The first is abstracted as a kind of "template" in order to recognize further experiences as similar or equivalent and, eventually, in order to form a concept to which the word "apple" may be associated. In Piaget's terms, this is an *empirical* abstraction. The unity or thinghood, on the other hand, must eventually be abstracted in order to obtain a concept of "thing"; but there is something else which usually takes place before: the construction of *plurals*. The child who is able to recognize an apple as an "apple", may already have learned, or at any rate will soon learn, the meaning of "apples". This may seem trivial, but the conceptual construction of the plural -- any plural -- is a separate task. Approaching a table on which there are several apples, a child who can recognize an apple-experience and has a name for it, could say "apple", "apple", ... as many times as there are apples on the table, but it could not say "apples". In order to use a plural, a different procedure is needed. What differentiates experiencing one apple from experiencing several, derives neither from the sensory signals that constitute the "apple" template, nor from the fact that each of the items is given "thinghood". The plural refers to another aspect of the experience, namely to the fact that the apple template was used *more than once*. That is to say, it refers to something the subject does, or has done, and unless the subject takes into account its own activity, there is nothing to associate with the plural form. The ability to discriminate and use one's own actions is what Piaget calls *pseudo-empirical* abstraction. It is not reflection in Locke's sense, but it is an indispensable step towards it.

It makes possible a new kind of abstraction, abstraction of a mental activity rather than of a sensory-motor characteristic. If a child has mastered the plural "apples" and then spontaneously says "dogs" or "mouses" when, for the first time, facing several of these animals, it has not only abstracted the crucial feature of repetitive recognition from prior apple experiences, but also associated that abstraction with the terminal "s" and can now attach it to other words. In Piaget's terms that would be a reflective abstraction, not in the sense of conscious mental reflection, but rather as in optics, because what was abstracted in the original experiences with apples is now reflected by, or projected onto, a new situation. In order then to arrive at what Piaget calls reflective thought the child would have to reflect upon and compare several "reflective abstractions", notice what it is doing, and thus realize the principle: If there is more than one of a thing, add an "s" to the appropriate word.

Clearly such a reflective thought is a far more sophisticated achievement than merely adapting a procedure that has proven successful to a novel situation. Reflective thought, above all, requires the ability to re-present to oneself past experience, not only as compounds of sensory signals but integrally, including one's own actions and contributions. To remember and re-present to oneself an activity one has carried out, is one thing; it's quite another to reflect upon it and attend to how it was composed. To enact something in the re-presentational mode is like a play-back: one is in it and, in a very real sense, one acts again. Reflecting upon it requires detachment, and the detachment is achieved by using symbols rather than the actual records of acts. The use of symbols, Piaget has always insisted, is what turns actions into operations -- and reflective thought is decidedly operative and employs symbols whose concatenation in symbolic re-presentation implies action but is itself not action. That explains why, in re-presentation, complex patterns and sequences of acts can seem instantaneous: one word may stand for a year's experience. (Piaget, 1945)

How does this relate to numbers? Well, number words are unquestionably symbols. If I now presented you with a five-digit number, you would surely understand what I am saying. But what does "understanding" mean in this case? I doubt that you would re-present to yourself a truck-full of apples or any other collection of objects. Steeped, as you are, in "Mathematics Education", you might translate the number into so many "cubes", "flats", "sticks, and "units", but no ordinary human being would do that -- yet they would all understand the number word. I suggest that the ordinary way of understanding here means no more and no less than that the listener recognizes the words as belonging to the conventional number word sequence and registers them as such. Number words are somewhat special symbols for the reason that they belong to a homogeneous verbal system that has its own fixed rules of composition. In that respect they are different from the words of all the other semantic domains that are self-contained, e.g., that of color terms. All number words are "transparent" to anyone who knows the composition rules. Consequently, any element of the number word sequence provides immediate access to the results of several abstractions, provided these abstractions have been reflected upon and thus made available independent of any particular experiential context. Thus, hearing, for instance, the words that go with "27,272", you *understand* the utterance because you are aware that you could at once retrieve any of the following abstractions:

a) a *collection* of so many units, no fewer and no more;

b) because the collection is bounded in that sense, it can itself be treated *as a unit*;

c) how *many* units are contained in the indicated collection can be determined by *coordinating one unit with each of the number words that precede the given one* in the number word sequence;

d) the collection can be constituted also by using *other kinds of unit* that are, in fact, indicated by the systematic form of the symbol, namely: $(2 \times 10^4) + (7 \times 10^3) + (2 \times 10^2) + (7 \times 10) + 2$.

To experienced calculators other "facts" may be equally accessible; e.g., that the indicated collection is a composite also of other "units", namely 4 x 6818 or 14 x 1948. But these insights would not be considered necessary in the way the composition by "tens" and "hundreds" is. Ten and hundred are standard units in our decimal system and they are used all the time; therefore, children must learn to handle them.

Although "ten" is used as an abstract counting unit like "one", the way in which it must be abstracted is not quite the same. While the basic unit concept can be abstracted from sensory-motor objects, such as apples and dogs, there are no sensory-motor objects from which to abstract the concept of ten. There are only collections that happen to comprise ten units, but their ten-ness has to be discovered by counting. When children learn to recognize apples, the attentional pattern that makes each apple a discrete object is inherent as an unconscious contribution of the experiencer. In contrast, when children look at the outstretched fingers of both their hands, they indeed constitute each finger as a discrete unit, but that these fingers require ten instantiations of the attentional unit pattern can be established only by counting.* To conceive ten-ness, then, needs abstraction from a self-generated activity and reflection upon that abstraction. Only when it has entered the realm of reflective thought can "ten" become available as the complex concept of adult usage. Apart from being a number word among others of the standard sequence, "ten" must be understood as a symbol that can be interpreted, on the one hand, as a collection of so many units and, on the other, as a unit in its own right which can be iterated and counted (Richards & Carter, 1982). This dual function is different from the dual function, or ambiguity, of certain words such as "bat", whose interpretation is guided by the linguistic or situational context; in the case of "ten", both interpretations have to remain accessible in every context, because when counting tens by coordinating them with "one, two, three, ..." the counter must remain aware of the fact that his count is equivalent to another that would go "ten, twenty, thirty, ..."

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* Children may, of course, recognize patterns of five fingers and learn by simple association that two hands make "ten" -- but this would not, without counting, lead to a concept of ten consisting of ten units.

Acquisition of the counting units "ten", "hundred", ... thus requires more than the ability to count in tens or hundreds. It requires the complete abstraction of these complex concepts from any particular counting context. This is a demanding reflective task, quite similar, I believe, to that which has to be achieved for all other concepts before they become available to us in spontaneous re-presentation. The child that cannot readily shift back and forth from "ten" as counting unit to "ten" as collection is in a position similar to that in which we all find ourselves with regard to a good many words: though we know their meaning well enough when we hear or read those words, they are not available to us when we speak or think.

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THE NUMERATION SYSTEM

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ABSTRACT

This paper explores the conceptual basis of children's understanding of the Numeration System. It reports some findings about children's profound lack of understanding, and some speculations about possible sources of their difficulties, based on an extension of the idea of "unit".

The Numeration System, including such features as place-value, multi-digit notation, and the concept of 'base', depends on repeating structures. The construction of a number as a composite of tens and ones requires two conceptual advances which extend radically beyond both mastering the verbal and written number sequence, and constructing number as an abstract unit item. The child must first construct a new iterable unit of ten, and then this new abstraction must be coordinated with the already existing iterable units of ones.

INTRODUCTION

From the second through the fifth grade, many children struggle with arithmetic problems that result directly from their failure to understand the structure of the Numeration System. It is certainly obvious, and has been often noted, that the Numeration System, including such features as place-value, multi-digit notation, and the concept of base (as in "Base-Ten"), depends on repeating structures.

In this paper we investigate the nature of the conceptual development that enables children to integrate these various repeating structures, and to genuinely understand the Numeration System. This underlying coordination is necessary for the meaningful mastery of ideas such as:

1. the Base-Ten structure of our number system,
2. the notational system,
3. the counting sequence,
4. carrying and borrowing, and
5. multi-digit multiplication and division.

These are all related, and attempts to teach, or study, them independently, while possibly meeting with local success, ultimately fail. Understanding the Numeration System involves much more than acquiring competence in some selection of the above skills and concepts. Such skills, if children do acquire them, are often only surface manifestations and may mask a lack of understanding at a deeper level.

As surface features, there are several forms of repetition within the number sequence -- the continued repetition of the number words from "One" to "Nine" within a decade structure, and the decade structure itself which repeats this number sequence. This repetitive structure exists in both counting and in written numerals.

Many children, by the third grade, have mastered the repetitions of the number sequence in both counting and written numerals. But these are largely verbal and visual pattern skills. The conceptual mastery of the Numeration System to One Hundred, must occur at a deeper level.

The construction of a number as a composite of "Tens" and "Ones" is an extremely abstract, and difficult, procedure. It differs significantly from constructing a number as an abstract unit item, or even constructing a number as the sum of two, or even three, other numbers.

A child's ability to construct, conceptually, a number over Ten out of "Tens" and "Ones" signifies the construction and integration of different kinds of units -- a new iterable unit of "Ten" must be constructed, and it must be coordinated with the iterable unit of "Ones".

The Numeration System to one hundred is based on a double repeating structure -- repeatable, or iterable, Tens, and repeatable, or iterable, Ones. For a child to understand this structure they must not only construct Tens as iterable units, but this construction must be coordinated with the construction of Ones as iterable units.

The distinction between "Ten" as a number, and "Ten" as an iterable unit is marked, linguistically, by the distinction between constructing a "Ten" as an abstract unit item, and constructing "Tens" as iterable units. The former is a specific number -- the result, for a particular child, of a reflective abstraction, the latter designates a category whose instantiations are producable.

ONES AS ITERABLE UNITS

The construction of "Ten" as an iterable unit follows the construction of "One" as an iterable unit. In this section we examine the evolution of "One" as an iterable unit. In presenting this, we briefly review some of the earlier work of the Georgia project, which focused on counting.

Counting is the production of a sequence of number words such that each number word is accompanied by the production of a unit item (see Steffe, von Glasersfeld, Richards and Cobb, 1982). In the earliest forms of counting, children have not constructed number. The unit item must be composed out of a sensory-motor experience -- an object, a figural image, or a movement. The significance of each number word is tied to the immediate context in which it is produced and the specific unit item to which it is associated. Moreover, in counting, the individual acts are not connected, that is, there is not a sense of producing "another and another and another". There is not a recognition that in a single counting action each counting act is separated by a single unit item. "One" is not a part of each counting act.

At the final sensory-motor stage, Counting Verbal Unit Items, the nature of the unit item being counted becomes much more arbitrary. A child will count extending fingers, swinging motions of the leg, nods of the head, and so on. The child appears to realize tnat the precise nature of what is counted is not relevant. This is seen in a curtailing of the counting act. For example, a child who is asked to add 12 and 4, begins by counting quite rapidly to twelve. The elaborate structure of the counting actions: uttering a word, constructing a unit item, and indicating the item, has been curtailed, or reduced, to simply uttering the word.

For the Counter of Verbal Unit Items, the unit item has become an iterable entity, not tied to a specific context. The unit item has been stripped of its distinguishable sensory motor features. Each number word, in a counting sequence, is separated, for the child, by a unit item -- an iterable "One".

This evolution of "Ones" as iterable units allows the child to curtail the counting act. In counting, the child produces "another and another and another", where there are no individual differences between units. This iterated unit has been stripped of its qualities. And the child is now finally ready for the construction of number as an abstract entity.

COUNTERS WITH ABSTRACT UNIT ITEMS

This next step, counting Abstract Unit Items, occurs when a number, like Seven, can be experienced, through reflective abstraction, as a composite unit made up of iterable "Ones". At this stage, the child can construct number. Eight, Nine and Ten are experienced as composite units. They are units, and they are also decomposable into units. This double abstraction produces numbers -- as units of units -- and it provides the child with the reasoning power of the number system.

A counter of Abstract Unit Items can experience a two digit number, like 23, as a composite unit. It is a unit of units. For that child, the number has a double significance -- as a single unit, and as a unit constructable from iterated units of One.

Even though a child may be able to construct Ten as a number, that abstraction is not adequate for the construction of the Numeration System to One Hundred. Moreover, even more sophisticated developments on the part of Abstract Unit Counters still precede the construction of the iterable "Ten".

The child still needs to construct relationships between numbers. For example, the child must be able to construct 23 as made up of 15 and 8. Eventually, such a child may even construct addition and subtraction as reversible operations, and may know, without counting, that since 15+8 = 23 then 23-8=15. This is a real "part, part, whole" relationship -- both parts are seen as wholes -- as abstracted numbers.

TENS AS ITERABLE UNITS

In spite of the very real sophistication of the child at this point, our evidence is that neither mastering the surface notational or linguistic repetitive structures, nor being an abstract unit counter, nor constructing the reversibility of addition and subtraction enables children to fully master the Numeration System to One Hundred.

Even when a child can construct 23 as made up of 15 and 8, she may still be dealing with 15 and 8 as individual units. It is not until such units can be used as iterable units -- considered not only as numbers, but also as things that can be easily counted as units -- in the construction of other numbers, that the necessary conceptual basis for understanding the Numeration System exists.

Before children can use "Ten" as an iterable unit to construct larger numbers, they must be able to strip it of its composite quality to make "Tens" -- just as "One" had to be stripped of its sensory motor qualities before it could be used to construct larger composite units. Like the unit "One", "Ten" must be transformed to be seen as a primitive in the system, so that it may be iterated as "another and another and another."

Seeing ten as iterable is distinct form being able, say, to add ten and ten to make twenty. Seeing twenty as built up out of two units of ten is conceptually different from simply being able to add ten and ten to get twenty. In this sense, "Ten and Ten" are distinct from "Two Tens". The former is not different from taking any pair of numbers -- say Fifteen and Eight.

There is a special relationship which the child may generate between 15, 8 and 23. But this is different from the relationship between, say Two Tens and Three Ones, and 23. In the former case there is a triadic relationship between three numbers. In the latter case there is a relationship between a number, say 23, and a structure "Two Tens and Three Ones". Thus, we also distinguish this structure from "Ten and Ten and Three".

There is a second conceptual leap to being able to conceive of a number as made up of two different kinds of units. The use of "Ten" as in iterable unit must be coordinated with "One" as an iterable unit, so that a number can be constructed using both, conceived as made up of two different kinds of units.

As 23 took on a double significance for an Abstract Unit Counter "Ten" must now have a double significance, as an iterable unit, and as one among at least two iterable units used in the construction of numbers.

RESEARCH ON NUMERATION

Because the difficulty of this new double abstraction is often not seen, the extent of the problems with numeration are often underestimated. Lauren Resnick reports that in one of her studies only one child in the third grade could identify the "9" in "96" as standing for Ninety, or Nine "Ten-blocks". Our own experiences more than confirm this.

In an informal second grade teaching episode, Carter focused on the sort of connection the children made between the symbolic representation of a two digit number, and its grouping structure. The children were given a series of shoeboxes filled with different numbers of loose interlocking cubes. Each box was labeled with the number of cubes it contained.

The children were then told a story about a candy factory, where they made candy sticks Ten cubes long. The problem was to see how good they could get at figuring out how many "Ten sticks" could be made from each box, and how many extra Ones would be left over. They could assemble the cubes in each box into Ten sticks if they needed to check their thought. They were then to record on a separate sheet their findings of how many Tens and how many left over Ones. They were also told that the numerals on the boxes told how many were in each box, and that they should use those numerals to identify each box on their record sheets.

The children seemed intrigued with the task, and went diligently to work. However, not a single child connected the numerical labels on the boxes which they could all read (e.g. have you done "Thirty-One" yet) with the number of Ten sticks they constructed. Finally, near the end of the work session one girl said, "Oh, I get it. The number tells you how many sticks there are."

In a recent, more formal, study (Kamii and Carter) Seventy-Seven first through sixth graders were asked to count the number of chips in a pile (e.g. 23). They were then asked to write the appropriate number. Finally, they were asked to relate each of the digits in the two place number they had written to the pile of chips, .e.g. "Show me what the "2" in "23" stands for with the pile of chips." In this study fully half of the sixth graders were unable to show that the "2" in the "23", they had correctly written, stood for two groups of Ten chips.

Although these children could count accurately, could write the correct two digit number, and many could even tell us, when asked, how many Tens and Ones were in a number, they failed to apply that knowledge to the task of structuring a group of objects according to our Numeration System. They could not construct numbers appropriately from their knowledge of number words or written symbols.

The difficulties experienced by these groups of children relate directly to their inability to use Ten as an iterated unit, as an abstracted repeatable unit with which to count and construct other numbers. And at a second level, their inability to integrate the use of Ten as an iterable unit with the iterable Ones.

SUMMARY

From a conceptual analysis, and preliminary studies with children, we have argued that there are several developments which clearly precede the child's understanding of the Numeration System.

A child may have constructed number -- in our terms be a counter of abstract unit items.

A child may also have constructed numbers as composites of other single numbers. This may even extends to the construction of the reversibility of addition and subtraction.

A child may have mastered the repeating number sequence of both counting and writing numerals.

None of these accomplishments, as sophisticated as they may seem, are adequate for an understanding of the fundamental structure of the Numeration System. What is necessary is a reconceptualization of Ten as an iterable unit, abstracted to a point where it can be used as a unit to construct other numbers.

Finally, one might ask where in the curriculum we might look for the roots of this development. Our analysis suggests that rather than the repeated surface structures of the Numeration System we might better look for developmental beginnings in processes like multiplication.

It may well be that Ten is not the second iterated unit. Many children could gain inportant experience working with other numbers -- two's, three's, or fives for example -- in work with multiplication. In fact, we suspect that work in multiplication could be easily extended to provide a fuller understanding of numeration.

In closing, we suggest several extensions of the ideas in this paper. Rather than limiting our focus to the Numeration System, an analysis based on the idea of an iterated unit might form a basis for children's understanding of a range of mathematical topics. These would include, not only numeration, but also such things as multiplication and fractions.

STAGES IN THE CONSTRUCTION OF THE COUNTING SCHEME[1]

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Based on analyses of behavior of six children who were students in a two-year teaching experiment we identify four periods and two stages in the development of the counting scheme. The observed periods are the perceptual period, the motor period, the verbal period, and the abstract period. Those four periods are further subdivided into two stages — the sensory-motor stage and the abstract stage. The former stage consists of the perceptual, the motor, and the verbal periods. The prime behavioral difference between the two stages is the flexible use of counting in the abstract stage. The three children who reached that stage could form abstract composites of both actual and intended counted units. The children who remained in the sensory-motor stage were restricted to forming temporally or spatially bounded composites of experiential items.

A model for the construction of counting was presented at the Counting Types Symposium held in April of 1981 at the University of Georgia (Steffe, von Glasersfeld, Richards, & Cobb, 1982). The essence of the model is the five types of unit items children create while counting. We make a sharp distinction between what a child intends to count and what it actually counts in the following summary of these counting types. This distinction was approached by Judd in 1927.

> One recognizes a group of objects as composed by many individual items only when one points to each one or otherwise analyzes the group by reacting in succession to each member of the group. Until a person has reacted to each member of a group . . . the group will appear in his consciousness as a vague and ill-defined mass of experience (p. 50).

Analyzing the "vague and ill-defined mass of experience" into constituent unitary wholes is involved in counting activity. A child normally establishes a collection of perceptual items prior to counting and then counts the items of the collection. While counting, each perceptual item is, in turn, considered anew as a unit item. We call this counting perceptual unit items.

A manisfestation of an independence from immediate perception when creating a countable item in an act of counting occurs when a collection of perceptual items is not within the child's range of perception or action. In this case, the child might coordinate the sequential production of number words with the sequential production of a visualized image of the perceptual item. In this case, we would say that the child counts figural unit items.

Motor acts (movements) of the child become countable items when the counter becomes aware of both the unitary character of the individual motor acts and of their coordination with figural items or with actual perceptual items. The essential feature of counting motor items is that the child uses the motor act as a substitute for the perceptual item or its figural representative. This substitution for the first time provides the child with independence from its immediate perceptual world in counting because motor acts can be produced at will.

The next step removes counting even further from the perceptual items. An act of counting always involves the coordination of two productive activities -- the production of a unit item and the utterance of a number word. When the utterance comes to stand for a concomitant unit item, the utterances themselves carry the significance of countable items. In this case, we say that the child counts verbal unit items.

Children who can count abstract units can take any one of the sensory motor units described above itself as a unit. This capacitv to make units of units leads to the possiblity of considering anythingwhatever as a countable unit. These children can formulate the intention of finding out how many times they are going to count before they actually create countable unit items.

We categorize children as being counters of a particular type if they are observed counting items of that type but not items of a more sophisticated type except in special contexts. The primary problem that we address in this paper is whether these counting types constitute stages. The various counting types do provide a qualitative classificatory scheme which lends itself to the stage question. We address three criteria for stages -- the invariant sequence criterion, the integration criterion, and the modification of structure criterion (Flavell, 1963).

In our study of the stage question, we conducted a teaching experiment that started in October of 1980 and ran until June of 1982. The six students in the experiment were all six years of age and in the first grade at the outset. We taught each child twice weekly in teaching episodes lasting anywhere from 10 to 30 minutes. These teaching episodes were viewed as being "enrichment" by the classroom teachers. Each episode was video taped and our analysis of the children's behavior is based on this large corpus of video tapes. The teaching experiment lasted for approximately 16 weeks for each of the two school years.

Three of the children, Brenda, Tarus, and James, were counters of perceptual unit items at the start of the teaching experiment. The three other children, Jason, Scenetra, and Tyrone, were counters of motor unit items. James was a substitute.

OBSERVED PERIODS

Figures 1 and 2 are graphical presentations of the periods that we observed.

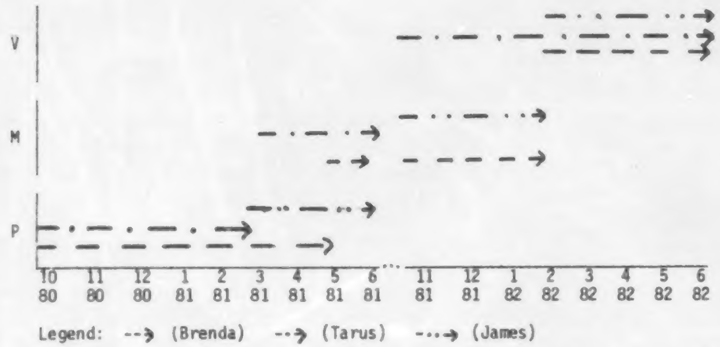

Figure 1: Periods in the development of counting: Brenda, Tarus, and James.

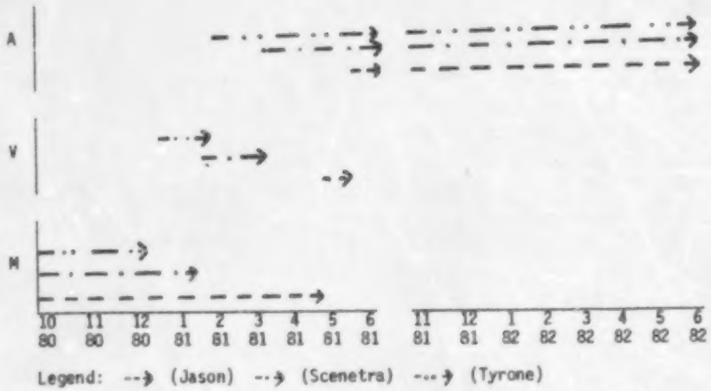

Figure 2: Periods in the development of counting: Jason, Scenetra, and Tyrone.

There are several noteworthy general observations:

1. The three children who were counters of perceptual unit items at the outset of the experiment never passed through a figural period. We observed Tarus and James counting figural unit items, but that was never a general characteristic of their counting behavior.
2. The verbal period was a transitional period for the three children who were counters of motor unit items at the outset of the experiment. For the three other children, it was a major period that lasted at least four months.
3. The three children who were counters of perceptual unit items at the outset of the experiment never entered the abstract period. In other words, these three children, in spite of our best efforts, never constructed number during their first and second years of formal schooling. This was the single most important result of the teaching experiment.
4. There was an invariant sequence in the emergence of counting types for the two groups of children -- perceptual, motor, and verbal for the three children who were counters of perceptual unit items at the outset; and motor, verbal, and abstract for the three other children. The lengths of the periods varied, but not the sequence of emergence.

We will consider two explanations of why the children who were counters of perceptual unit items never entered the abstract period. If the motor acts for these three children were not substitutes for perceptual items or their figural representatives while they were in the motor period, counting their motor acts would be on par with counting perceptual unit items. That is, the children would count motor acts per se rather than motor acts as substitutes for other items. The motor acts would lack the quality of being the "reflection of an action into representation when that action is interiorized" (Piaget, 1980, p. 90). The crucial aspect in this optical characterization of reflection is whether forming motor unit items in counting is the result of their isolation in the context of co-occurance with figural representatives of perceptual items.

The second explanation of why these three children never entered into the abstract period concerns that aspect of reflective abstraction which "involves 'reflexion' in the sense of a mental reorganization" (Piaget, 1980, p. 90). We see three levels of mental reorganization regarding counting. The first consists of those reorganizations of which the child is unaware; this level of reorganization does not involve re-presentation. The second level of reorganization involves the re-presentation of counting activity in the child's immediate past experience. Here, the child may view a counting activity from an alternative perspective after the activity has been carried out. The third of reorganization involves the re-presentation of counting before the activity is carried out. The child views the activity from two perspectives prior to actually counting. Perhaps enterance into the abstract period requires reorganization at the third level.

We turn now to a brief analysis of Brenda and Tarus, focusing on their progress within the periods observed. Our goal is to isolate possible factors for their not entering the abstract period. We omit James only because of a lack of space.

PERCEPTUAL PERIOD

Brenda

The major advance Brenda made in the perceptual period was to make finger patterns when collections were perceptually inaccessible. For example, in the context of finding how many marbles were hidden beneath two cloths, one hiding six and one hiding three, she first simultaneously put up five fingers on her right hand and one on her left hand. Next, she put up two more fingers on her left hand, completing a finger pattern for "three". Finally, she counted all eight of her extended fingers. Her failure to make a separation between the finger pattern for "six" and those she subsequently put up indicated that she relied on a visual separation. As Brenda had to make these finger patterns before she counted, it can be inferred that this advance did not involve a reorganization of her counting scheme. She counted fingers per se.

Tarus

The first major advance that Tarus made towards freeing himself from immediate perception was to develop spatio-motor patterns through "five". For example, to solve a task where three items were hidden by one cloth and four by another, Tarus would utter "1-2-3" in synchrony with tapping on the cloth where his points of contact formed a triangular three, and then continue tapping on the other cloth in synchrony with uttering "4-5-6-7", where his points of contact this time formed a square four. Here, there was a first level reorganization of his counting scheme in that he had coordinated schemes for the production of spatial patterns with his counting scheme. Use of this scheme was restricted to contexts in which he could substitute specific spatial patterns for both collections. If a screened collection was beyond the range of his patterns, he was stymied. For this reason, Tarus was still classified as a perceptual counter.

Tarus' second major advance in the perceptual period was to substitute finger patterns for hidden perceptual collections. He immediately sequentially put up fingers to establish finger patterns, but failed to make separations just as did Brenda. Eventually, Brenda also sequentially put up fingers but she, too, counted her fingers rather than the acts of putting them up.

Both children were still in the perceptual period, but there were coordinations with spatial or finger patterns. When Brenda finally put up fingers to establish finger patterns, her fingers were not substitute items for the items of the hidden collections. She was counting her fingers per se. Tarus, however, provided every indication that his fingers were substitutes (perceptual) for the hidden collections because he would oscillate between looking in the cup which contained say, marbles and looking at his fingers.

MOTOR PERIOD

Brenda

Putting up fingers sequentially to establish finger patterns provided an occasion for Brenda to notice the motor acts. She finally made double use of her fingers while counting. For example, when counting to find thirteen and four more, she sequentially put up all ten fingers, closed one hand and seqentially put up three fingers for a second time while uttering "1-2-. . . -13". As she was not directed to reuse her fingers in this way, we infer that she counted the motor acts of putting up fingers. But she continued to have a very difficult time in making and maintaining separations between her initial counting acts and their continuation. The basis for her difficulty was a lack of spatial and temporal patterns. She relied solely on finger patterns.

Tarus

Tarus' advancement to the motor period was facilitated by his abbreviation of spatio-motor patterns.

> T: (Puts eight and then four more marbles into a cup. To find how many in the cup, he sequentially put up eight fingers) 1-2-3-4-5-6-7-8. (He then holds the three extended fingers on his left hand with his right hand while putting up his left thumb and little finger. He then puts up two more fingers of his right hand) 9-10-11-12. Twelve.

Tarus put up two fingers for a second time when he counted "11-12". This is prime indication that he was not counting his fingers as perceptual items. He now took the motor act of putting up fingers as countable items.

The two primary differences between the two children were (1) the function of putting up fingers, and (2) the anticipation of continuing to count. Putting up fingers for Brenda was a carry over from establishing finger patterns. She intended to count her fingers, but in the activity of counting, she counted the motor acts of putting them up. The "distance" between *what* she *intended* to count and *what* she *actually* counted was much less than it was for Tarus. He *intended* to count perceptual items other than fingers and *actually* counted the motor acts putting up fingers.

Tarus could anticipate continuing to count beyond an already completed counting activity (cf. protocol) because "four" referred to a number word sequence -- a temporal pattern -- consisting of *any* four sequential number words that he knew. The only patterns to which number words referred for Brenda were finger patterns. Consequentially, she could anticipate continuing to count only in those cases when she intended to establish finger patterns.

VERBAL PERIOD

Brenda

In late January of 1982, there was indication that Brenda intended to count her motor acts rather than perceptual items. To find eleven and three more, she tapped her forehead while synchronously uttering "1-2-...-11" and then continued "12-13-14", sequentially putting up three fingers. Shortly after this (in early February of 1982), she curtailed the coordination of a motor act with the utterance of a number word when counting the first of two collections.

In February of 1982, Brenda could finally re-present counting activity in immediate past experience and reorganize the activity. This reflection led to a contextual advance in her use of counting. Upon being directed to count by her teacher, Brenda first uttered "1-2-. . . -24" when solving "24-6". She then simultaneously put up six fingers and, with prompts from her teacher, she counted "24-23-22-21-20-19". Having just uttered "1-2-. . . -24", she re-presented this activity, reorganizing it as counting backward. This was the first indication that she could view her counting activity from an alternative perspective while solving a task. But it was crucial that she actually be directed to count. She could not spontaneously solve subtraction tasks in this way. She could not reorganize a re-presentation of counting before she actually carried out the activity. Counting in the context of missing addend tasks was also a result of her capacity to reorganize a re-presentation of counting activity in her immediate past experience.

Tarus

In November of 1981, Tarus counted two hidden collections of nine and six by subvocally uttering number words. While counting, he sat silently for 12 seconds and then responded "fifteen". He re-enacted his solution by uttering number words aloud ("1-2-. . . -15") in synchrony with putting up fingers. Tarus always started counting at "one" and never intended, before he started to count, to keep track of how many times he was going to count. This is shown by his lack of anticipation in solution of missing addend tasks.

He did not use counting to solve missing addend tasks where he had not previously counted part of the collection. Like Brenda, he could reorganize a re-presentation of already completed counting activity. For example, Tarus counted seven blocks which were then hidden. Some more were put with them and he was told there were eleven. He then rapidly uttered "1-2-3-4-5-6-7", paused, and then continued "8-9-10-11" and knew the answer was "four". This solution occured in a context where he could view the activity of counting to "eleven" as a continuation of his already completed count to "seven". Normally, he could not solve these tasks.

Throughout the remainder of the school year (to June of 1982), neither Brenda nor Tarus viewed counting from two perspectives prior to counting when solving subtraction and missing addend tasks. They never intended to keep track of how many times they counted before they started counting.

These case studies indicate that reflection in the sense of the optical analogy clearly is not the crucial determiner for why some children but not others construct number. Such reflection seems crucial, however, for children to enter the verbal period. Moreover, both Brenda and Tarus displayed reorganizations at the second level. But these reorganizations were not sufficient for them to enter the abstract period. To shed further light on what it takes to enter the abstract period, we provide a brief analyses of Jason and Scenetra. Tyrone is omitted because he had the shortest motor and verbal periods of these three children.

JASON

The motor period

Early in the motor period, Jason used two distinct counting schemes, his pointing scheme and his finger extension scheme. He used his pointing scheme to count partially hidden collections. Starting with "one", he could first count the visible portion and then continue to count over the screen hiding the rest of the perceptual items. He took acts of pointing as countable items in the continuation of counting. The pointing acts were clearly substitutes for the hidden perceptual items.

Jason used his finger extension scheme when both portions of the collection were screened. Here, his search for sensory content led to the intention of counting his fingers. But in the activity of counting, he took the motor acts of putting up fingers as countable items just as he had constituted his pointing acts as countable items in the activity of counting.

In February of 1981, Jason could re-present counting perceptual items and isolate the motor activity in counting before counting. For the first time, he could use his pointing scheme to count two hidden collections. For example, in one task involving seven marbles hidden in one cup and four in another, he uttered "1-2-. . .-7" while synchronously pointing in the direction of the first cup and then continued "8-9-10-11" while synchronously pointing in the direction of the other cup. He was sitting about two meters from the cups and knew what he was going to do before counting.

Jason's most sophisticated counting solution in the motor period occured in the context of solving a task where four of a row of ten poker chips were hidden and he was directed to count to find how many were hidden after being told there were ten in the row. First, Jason counted the visible chips and then continued "7-8-9-10" while synchronously pointing over the cloth.

> T: How many are under there?
>
> J: (Not knowing, he recounts, pointing to each visible chip) 1-2-3-4-5-6 (again, points over the cloth where his points of contact form a row) 7-8-9-10. (Still not knowing how many are covered, he sequentially puts up fingers) 1-2-3-4-5-6-7-8-9-10. Five (holding up an open hand).

Jason re-enacted his initial count because he did not know how many chips were hidden. However, he had no visible records of counting, which led him to count a third time, this time putting up fingers as substitutes for the screened items. This was a level two reorganization. His answer, "five", indicates that he did not reorganize his re-presentation of his last counting activity and view it as a count of two collections. Instead, he relied on the visual separation in his perceptual records.

Jason did not count to solve a missing addend task unless he was directed to do so while he was in the motor and verbal periods. There was no indication that he could reorganize a re-presentation of counting activity prior to counting.

The abstract period

Jason could now take the units he created either before or while actually counting as a composition of units -- he could now form composite unities of individual units. This is indicated by his spontaneous solution of subtraction tasks. For example, on one occasion he was asked how many marbles were left in a cup after five of nineteen had been removed. Jason counted "19-18-17-16-15" while sequentially putting up five fingers and said "there's fourteen in there"! He clearly intended to find out how many times he would count when he continued beyond an intial count of the five marbles. That is, he considered that his backward count of all nineteen marbles was also a count of two collections prior to counting. This was a level three reorganization. Stopping at "fifteen" indicated to us that he took the counting acts "19-18-17-16-15" as a composite unity. His finger pattern for "five" was there, but so was his realization that it was the implementation of a composite unity for "five". Stating so emphatically that fourteen marbles were in the cup without counting indicated to us that "fourteen" denoted the number word sequence "14-13-. . ..-1" which he never uttered and connoted the numerosity of the composite unity whose elements were these number words. He clearly took the marbles in the cup as having been counted.

SCENTRA

The verbal period

In January of 1981, Scenetra counted collections of five and three items by uttering "1-2-3-4-5--6-7-8". Her utterances carried the significance of countable items which could stand for the screened items.

While she was still in the verbal period, Scenetra could, in certain contexts, form composite unities from already counted items. On one occasion, three of a row of nine items were screened and Scentra was directed to count backwards to find out how many were hidden. First, she counted the visible items and then continued "3-2-1-zero" while pointing over the cloth. She was directed to count again. This time she sequentially put up fingers of one hand while synchronously pointing over the cloth as she continued "3-2-1" and then answered "three". Clearly, Scenetra put up fingers to record her acts of counting the screened items. Her extended fingers were more than records for the screened items. However, she formulated the intention of finding out how many times she would count after she had counted over the cloth once. This level two reorganization involved viewing her initial count as a count of two collections and taking at least the count of the second collection as a composite unity. While Scenetra formed a composite unity of already counted items, there was no indication that she could create composite unities prior to counting while she was in the verbal period.

The abstract period

In March of 1981 Scenetra spontaneously counted to solve the sentence 8 + =12. She counted-on beyond a finger pattern for eight, putting up fingers in synchrony with the utterances "9-10-11-12", and then picked up the felt numeral "4". Here, Scenetra realized that she could find out how many times she was going to count before she counted. She made a level three reorganization and created at least one composite unity from intended rather than actual counting acts.

DISCUSSION

Jason's and Scenetra's awareness of what they were going to do before they did it was also characteristic of Tyrone when he was in the abstract period. All three of these children could reorganize re-presentations of counting before they started to count. They could anticipate finding how many times they were going to count and deliberately kept track of their counting acts. This is in stark contrast to Jason's counting while he was in the motor period and to the counting activity of Brenda, Tarus and James that we observed. There is a profound reorganization of counting at the abstract level that we did not observe at the lower levels. Scenetra's context specific double counting while she was in the verbal period provides valuable clues as to the nature of this reorganization. Below we consider the various products of reflective abstraction at the three levels of reorganization of the counting scheme.

First (level one), there are collections of counted motor or verbal unit items such as those produced by Jason while in the motor period when he counted his pointing acts. These experiential collections are bounded temporally by the beginning and the end of the counting activity. When the motor or verbal items refer to location, there also can be spatial boundaries. Second (level two), there are re-presentations of these collections. These figural collections have the temporal boundaries of the re-presentation. Such re-presentations and subsequent reorganizations occured for Brenda and Tarus (as well as James) when they were in the verbal period, and for Jason and Scenetra (and Tyrone) when they were in the motor period. Third (level two), there are the composite unities that Scenetra formed from already completed counting activity when she was in the verbal period. Fourth (level three), there are composite unities that Jason and Scenetra (and Tyrone) formed before counting when they were in the abstract period. The new composition of counting acts can be now *what* is re-presented before a child begins to count. The child now engages in what, to the observer, appears to be *spontaneous*, *flexible* counting. The spontaneous counting activity is a prime indication of *reflected* abstraction. When the products of reflective abstraction is freed from any necessary relationships with material objects, "it gives place to what we may call reflected abstractions (products of reflective abstractions), and to a kind of reflective thought such as that which animates every formalization" (Piaget, 1980, p. 93).

Freeing counting from its material aspects was never accomplished by Brenda, Tarus, or James. They provided no indication of the capacity to make the third kind of product of reflective abstraction nor use it as material for further operating during their lengthy verbal periods. Scenetra, in contrast, created composite unities from already completed counting activity in specific contexts while she was still in the verbal period. (We took far fewer observations of Jason and Tyrone while they were in their short verbal periods). These contextual advances were precursor of the substantive reorganization that Scentra (and Jason and Tyrone) made when they entered the abstract period, as indicated by spontaneous, flexible solutions of a variety of tasks. For this reason, we take the structural reorganization criterion for stages as being satisfied. The integration criterion is satisfied for the periods -- children never lose their capacity to count unit items of any kind once the capacity is developed. Consequently, we take the abstract period as a stage in the development of counting. The three preceding periods (perceptual, motor, and verbal) constitute another stage that we call the sensory-motor stage.

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EFFECTS OF SIZE OF SET, HOMOGENEITY OF OBJECTS, AND COLLECTION NOUNS ON CHILDREN'S ACCURATE COUNTING AND USE OF THE CARDINALITY RULE

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Three- and four-year-old children were asked to count small (4,5,6) and large (9,12,14) sets of either homogeneous or heterogeneous objects and were then immediately asked how many objects there were. Use of the cardinality rule was inferred if the response to the "How many?" question was the last number word just said in the counting. Questions for half of the children used class nouns (e.g., blocks), and those for the other half used collection nouns (e.g., blocks in the pile). Size of set, homogeneity of objects, and verbal label (class or collection terms) had no effect on use of the cardinality rule, contrary to results reported by Gelman and Gallistel and by Markman. Size of set had a large significant effect on accuracy of counting, but homogeneity of objects and verbal labels had no effect. Contrary to the positions of Schaeffer, Eggleston, and Scott and of Gelman and Gallistel, accurate counting did not precede the use of the cardinality rule developmentally. These two skills were independent for small sets (4,5,6) and use of the cardinality rule preceded accurate counting for larger (9,12,14) sets.

One of the many things that a young child must come to understand about numbers is that the last word said in counting a group of objects tells the number of objects in that group. Schaeffer, Eggleston, and Scott (1974) termed this understanding "the cardinality rule" and measured it by having children count arrays of from five to seven objects and then covering the objects and asking how many objects were covered. They gave this task and several others to children between 2-0 (2 years, 0 months) and 5-11 and proposed a developmental hierarchical sequence of the integration of number skills to form other number skills. In their hierarchy the cardinality rule was acquired shortly after age five and was formed by integrating pattern recognition of very small numbers (acquired before age 3) and the counting procedure (acquired just before age 4), i.e., accurate counting of small sets (5,6,7) preceded use of the cardinality rule.

Gelman and Gallistel (1978) discussed the cardinality rule using the term "cardinality principle." They concluded that the cardinality principle was acquired after the stable order principle (the knowledge that the number word sequence must be produced in a stable order) and the one-to-one correspondence principle (the knowledge that in counting objects a one-to-one correspondence must exist between the number words produced and the objects being counted). Gelman and Gallistel also found that with larger set sizes, counting accuracy and the use of the cardinality principle decreases markedly. They posited that children monitored their counting and when they were no longer accurate, they stopped using the cardinality principle.

Fuson and Hall (1982) presented an analysis of early number word meanings in children in which different meanings accrued to number words according to the mathematical structure of the setting in which the word was used. They proposed that when used in counting a group of objects, a number word takes on a counting meaning and refers only to the object with which it is matched in the act of counting. At the end of the counting act, therefore, a child must make a "count-cardinal" transition in word meanings from this counting meaning of the word as referring only to the last object counted to a cardinal meaning of that word as naming the number of objects in that group. Fuson and Hall made no conclusions about the effect of set size on the acquisition of this transition but it would seem that the transition might well be made all at once, i.e., that it would not necessarily depend upon set size. Fuson and Hall also argued that such a linguistic/conceptual transition by the child might be a somewhat low level one, especially at first, and it implied no understanding of cardinality beyond a rather vague notion that the cardinal meaning referred to the whole group and told "how many." Thus accurate counting (and the consequent accurate cardinality) is not required for the acquisition of this verbal meaning transition.

Markman (1979) found that using collection nouns (e.g., pigs in the family, blocks in the pile) rather than class nouns (pigs, blocks) resulted in a significantly higher use of the cardinality rule. She postulated that this was because the collection terms focused the child's attention on the set as a whole (e.g., on the family) rather than upon the separate objects (e.g., on the separate pigs).

The present study was undertaken for several reasons. In an earlier study in which young children counted groups of objects increasing in size from

4 to 31 (Fuson & Mierkiewicz, 1980), one of us had seen no indication that the cardinality rule depended upon set size or upon counting accuracy. Therefore we wished to examine these variables. We wished to use a more straightforward cardinality rule assessment than the covering procedure used by Schaeffer et al. We also wanted to replicate the Markman finding and to examine the generalizability of this finding to larger sets. Finally, we sought to assess the effect of another variable--the homogeneity or the heterogeneity of the objects in the array--on the use of the cardinality rule and on the accuracy of counting. It seemed possible that a homogeneous array might have opposite effects on these: by reducing the individuality of each object, it might focus the attention more on the array as a whole and thus increase the use of the cardinality rule relative to that displayed in a heterogeneous array, but it might make keeping track of where one was in counting a bit more difficult and thus might decrease the accuracy of counting relative to heterogeneous arrays.

DESIGN AND METHOD

The three independent variables of interest were: verbal label (class/collection terms), size of array (small/large), and type of objects in the array (homogeneous/heterogeneous), and the dependent variables were use of the cardinality rule and accuracy of counting. The verbal label manipulation was of necessity between subjects; all others were within subject variables.

Forty-eight middle-class children from a northern suburb of Chicago served as subjects. They were in the same age range as Markman's subjects: 3-2 to 4-7. Pairs were made from a ranking of the subjects by date of birth by joining the youngest to the next youngest, etc. One member from each pair was then randomly assigned to each verbal label condition. The mean age of each group was 4-0.

The small arrays were of size 4, 5, and 6; the large were of size 9, 12, and 14. Homogeneous objects were small free-standing dolls, small toy soldiers, and homogeneously colored 1" wooden blocks. Heterogeneous objects were small toy animals, heterogeneously sized and/or colored toy pigs, and heterogeneously colored 1" wooden blocks. The verbal labels used were "animals" vs. "animals at the party"; "pigs" vs. "pigs in the family"; "nursery school children" vs. "children in the nursery school class"; "soldiers" vs. "soldiers in the army"; and "blocks" vs. "blocks in the pile."

The procedure reported by Markman was used. For each problem, the objects were arranged in a line on the table in front of the child. Children in the Class condition were told, "Here are some nursery school children. Count the children." (The count errors were recorded but in this study by using the system in Fuson and Mierkiewicz, 1980.) "How many children are there?" Children in the Collection condition heard, "Here is a nursery school class. Count the children in the class." (Pause for a counting.) "How many children are in the class?" If a child counted again when asked, "How many are there?", the question was repeated once. Children were given the six small arrays first and the six large arrays seven (or, occasionally, ten) days later. Appropriate counterbalancing procedures were used.

The cardinality use score and the counting accuracy score for the three trials in each size by object type condition ranged between 0 and 3. A second scoring system with a range of 0-12 was also constructed for counting accuracy (4 points = no errors, . . ., 0 points = 4 or more errors). Three 2(verbal label) x 2(type of objects) x 2(size of array) ANOVAs with repeated measures for the last two factors were carried out, one on the cardinality rule scores and one on each of the accuracy of counting scores. A separate Markman replication one-way ANOVA on verbal label was carried out on the four trials in the small array condition similar to the four trials given by Markman.

RESULTS AND DISCUSSION

There were no main effects and no interactions in the main cardinality rule analysis. There was no hint of any effect of size of array (the means of the small and large arrays were 4.38 and 4.38 out of 6) or of type of objects (the means of the homogeneous and heterogeneous arrays were 4.40 and 4.37, respectively), while the nonsignificant effect of verbal label was in the direction opposite to that reported by Markman (class mean greater than collection mean: 4.97 vs. 3.80). Scores of individual subjects indicated that use of the cardinality rule was an all or nothing phenomenon: 44 of the 48 subjects had perfectly correct scores (12 out of 12) or perfectly incorrect scores (0 out of 12). Two of the remaining subjects had one error and one correct trial, respectively, and the final two subjects displayed consistent scores within but not across the two days of testing. The Markman replication analysis likewise revealed a nonsignificant effect of verbal label, with the means opposite to hers (class higher than collection: 3.28 vs. 2.56 out of 5, $\underline{F}(1,46)$ =

$p < .05$).

The source of the difference between the present study and Markman's study is not clear. Great care was taken to replicate Markman's study with respect to tasks and procedures including communication with Markman about these. A chi-square analysis on the total number of children in each study who failed to display the cardinality rule (by Markman's criterion of use on less than 3 of the 4 trials, 8 of 26 in Markman's study and 13 of 48 in the present study) revealed that these two study populations did not differ significantly in the number of children failing to show the cardinality rule ($\chi^2 < .1$); the children in the two studies who did not display the cardinality rule were just distributed in opposite ways across the verbal label conditions.

Analysis of the counting accuracy data using both attenuated and extended scoring scales revealed a large main effect of size of array, $F(1,46) = 45.53$, $p < .01$, and $F(1,46) = 45.16$, $p < .01$. Counting was much less accurate as the size of the array increased (per trial extended scoring means of 3.52 vs. 2.38 out of 4 and regular scoring means of 0.76 vs. 0.45 out of 1 for the small and large arrays, respectively). There were no significant effects due to verbal labels (class mean = 2.96 and collection mean = 2.94) or to type of object (homogeneous mean = 2.92 and heterogeneous mean = 2.97), and no interactions of any variables.

Table 1

Relationship between Cardinality Rule and Counting Performance

	Small Numbers		Large Numbers	
	Count Not Accurate	Count Accurate	Count Not Accurate	Count Accurate
Card Rule Not Accurate	7	7	11	1
Card Rule Accurate	16	18	27	9

Count Accurate and Cardinal Rule Accurate classifications based on criteria of 100% accuracy.

The numbers in Table 1 indicate that for small numbers (4, 5, and 6), counting accuracy and use of the cardinality rule are fairly independent skills: 23 of the 48 subjects showed one but not the other of these abilities. Counting accuracy clearly does not precede the use of the cardinality rule (16 vs. 7). A chi-square test for correlated proportions indicated that this difference is not significant ($\chi^2 = 3.52$, $\underline{df} = 1$, $\underline{p} < .10$), so neither of these skills significantly precedes the other. As the criterion for accuracy drops to 80% and then to 67% for both measures, subjects move to the right in both rows. The ratio of children who count accurately but do not use the cardinality rule to those who use the cardinality rule but do not count accurately becomes 9 to 12 and then 10 to 8. Thus these two skills remain independent even when a less stringent criterion for counting accuracy is used. With larger numbers (9,12,14), use of the cardinality rule clearly precedes counting accuracy. The ratio of children who count accurately but do not use the cardinality rule to those who use the cardinality rule but do not count accurately is 1 to 27 at a 100% accuracy criterion, and becomes 1 to 21 and 2 to 18 at 80% and 67% accuracy criteria, respectively. The chi-square for correlated proportions is significant for all of these ratios, $\chi^2 = 24.14, 18.18, 12.8$, respectively, $\underline{df} = 1$, $\underline{p} < .01$. Thus with numbers nine and larger, use of the cardinality rule quite clearly precedes accurate counting.

These results seriously call into question the Schaeffer, Eggleston, and Scott conclusion that accuracy of numbers just beyond the subitizing range precedes the use of the cardinality rule. These two abilities seem rather to be independent, with the developmental relationship being, if anything, more children using the cardinality rule without counting accurately than vice versa. The difference in the results of the two studies may lie in the procedure used for assessing the use of the cardinality rule. Schaeffer et al. covered their array before asking the cardinality question while we did not. Some attribute of the covered situation or a loss of memory of the last word said in counting due to a longer procedure seem to be possible sources of this difference.

There was also, contrary to the findings of Gelman and Gallistel (1978), no hint of an effect of set size on the use of the cardinality rule: the means for the small and the large arrays were not just nonsignificantly different, they were identical. Our procedure involved separate directions for the counting and the cardinality rule measures; children first were asked to count and were then asked how many objects there were. In the Gelman and Gallistel procedure, only one direction was given and this was sometimes a "How many?" question and sometimes a "Count them" or "Go ahead" directive. Perhaps as the experiment continued (the larger set sizes were given later), some children decided that the required task was primarily focused on counting accuracy and so they just counted and stopped telling the experimenter the results of their count on each trial.

The developmental accounts of the acquisition of the cardinality rule given by Schaeffer, Eggleston, and Scott and by Gelman and Gallistel, depending as they do on prerequisite accurate counting, clearly seem inadequate, at least as accounts meant to be universally true. The view that the cardinality rule requires a child to make a count-cardinal transition of meaning from viewing a context as a counting one to considering it as a cardinal one, as discussed by Fuson and Hall, seems reasonable in the light of the present results. The present study thus clarifies some points but does not tell us much about what triggers this count-cardinal transition in a child. The possible role of memory, suggested by Fuson and Hall, might be explored in future research: A child obviously cannot make a count-cardinal transition unless he can remember that last word just said in counting. Other factors, such as being told that the last word is a special one, might also be explored.

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THE DEVELOPMENT OF CHILDREN'S ADDITION CONCEPTS

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Based on analyses of behavior of six children who were students in a two-year teaching experiment, four stages have been identified in the development of addition concepts. The observed stages are the sensorimotor concept stage, preoperational concept stage, the operational concept stage, and the arithmetical concept stage. Several processes can be used to solve addition problems by children in each stage. The characteristic behavioral differences are: children in the lowest stage count collections of perceptual items; children in the second stage focus on counting sequences of actions; children in the third stage can construct composite unities; children in the final stage have habituated addition facts and use thinking strategies to find sums. Children with the same counting scheme may be in two different stages.

Concept formation is a slow developmental process. Children are expected to solve simple addition problems, for instance, 8 + 6, by "habituation" (Brownell, 1928, p. 121) yet the expectation is never fulfilled in some children. The position is taken that children's concepts are their own constructions rather than being externally transmitted (Piaget, 1976). The fact that children consturct their own concepts of addition is amply supported by research studies (Carpenter & Moser, 1982; Groen & Resnick, 1977) reporting the different strategies children use prior to and after recieving formal instruction. Counting is a basic process that children use in solving problems. Five types of unit items children create while counting have been identified (Steffe, von Glasersfeld, Richards, & Cobb, 1982). The five counting types, in order of sophistication, are counting with perceptual, figural, motor, verbal and abstract unit items. A child's counting scheme (Steffe, 1982) is directly influenced by the most sophisticated unit item the child can create and count. This paper analyzes the developmental stages in children's construction of addition concepts. The analyses are based on the behavior of six children who were students in a two-year teaching experiment. The study was started in October 1980 when the six students were all six years old and in the first grade. Each child was taught twice weekly for 10 to 30 minutes. The teaching episodes were all video-taped. The analyses are based on the large corpus of video tapes. The teaching experiment lasted till June 1982. Three of the children, Brenda, James, and Tarus were counters of perceptual unit items in October 1980. The other three children, Jason, Scenetra and Tyrone, were counters of motor unit items.

OBSERVED STAGES

Figure 1 is a graphical presentation of stages in the development of addition concepts observed in relation to counting schemes.

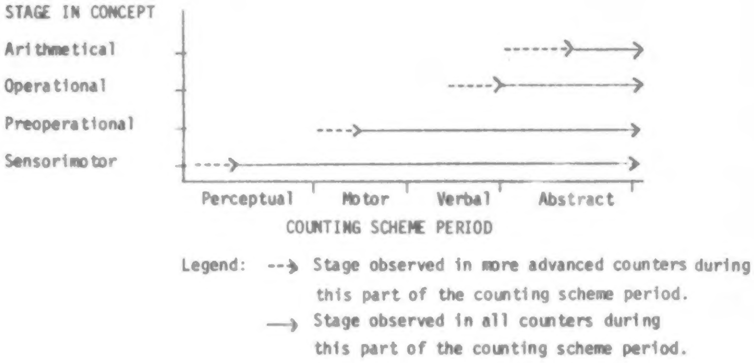

Figure 1: Stages in the development of addition concepts with counting schemes.

The following observations are noteworthy:

1. The two counters in the perceptual period in October 1980, Brenda and Tarus, had no method for finding how many items in two collections when either or both were screened.
2. Some counters in the same counting scheme period were observed in different but adjacent stages.
3. There was an invariant sequence in the developmental stages of the additional concepts for all the children.

4. At the end of the teaching experiment in June 1982; two children had reached only the preoperational concept stage, one child had reached the operational concept stage, two children had just entered the arithmetical concept stage, and one child had advanced in the arithmetical concept stage.

Sensorimotor Concept Stage

Children who solve addition tasks by counting perceptual items or their replacements are said to have a sensorimotor concept of addition. The children with this concept of addition can use several procedures or strategies to solve a simple addition task. The common and the characteristic behavior observed was that such children created and counted a single collection of perceptual items. The simplest strategy, Counting-All-With-Models (Carpenter & Moser, 1982) or Simple Counting (Brownell, 1928, Thaeler, 1981), was to construct two subcollections of perceptual items corresponding to the addends and then count all the items either after physically joining them or by making an enactive bridge from one collection to another. A slightly advanced strategy was to create and count perceptual items for both addends during a single counting activity. The difficulty in the latter strategy was for the child to know when to stop, that is, how to determine the exact items for the second addend. It was observed that children relied on the two figural patterns (von Glasersfeld, 1981), spatial patterns and temporal patterns to stop counting appropriately. Over reliance on spatial patterns, however, could lead to counting errors. For example, Tarus, a counter in the perceptual period, from October 1980 to March 1981, stopped counting inappropriately after extending seven fingers which completed a pattern for two on his right hand to solve "6 + 2" (presented with marbles in a cup). Tarus, however, successfully used spatial patterns to solve a task involving seven visible felt stars and six stars hidden under a cloth. The following is a portion of his protocol for the task.

T: Sequentially touches the visible stars in synchrony with uttering "1-2-. . . -7", then continues "8-9-10--11-12-13" (slightly pausing after 10) while touching six locations on the cloth, where his points of contact formed a triangular three pattern plus a linear three pattern.

The behavior of Tarus in the above task might appear sophisticated but a closer observation indicated that he counted sensory-motor patterns and was limited to patterns for numbers less than six. All the three children, Brenda, James and Tarus had sensorimotor concept of addition during their perceptual periods (cf. Steffe & Cobb, 1982).

Jason and Scenetra, counters in the motor period, had a sensorimotor concept of addition during the beginning of that period (cf. Steffe & Cobb, 1982). Jason used a strategy which involved a combination of motor and perceptual items in the same counting activity but carried the impact of counting perceptual unit items for Jason. To solve a task involving seven screened marbles and five visible marbles, Jason uttered "1-2-...-7" while synchronously pointing over the interviewer's hand (covering the marbles), then continued "8-9-10-11-12" while pointing at the visible marbles. Jason's solution is an example of a simple extension (Steffe, et al. 1982, p. 128) because it involved counting perceptual unit items after a pause in a sensory-motor counting activity. Jason was observed to use the same sensory-motor activity during the early part of his motor period. This indicated that his intention in the above task was to count a single collection of perceptual items.

Preoperational Concept Stage

Children who solve addition tasks by counting a sequence of motor acts or utter a sequence of number words are said to have a preoperational concept of addition. Children with this concept can therefore create and count motor unit items and verbal unit items. It was observed that actual (vocally expressed) counting activity was crucial to these children because they were unable to consider the number word, say "six", to signify a potential sequence of six motor acts or the number word sequence "one. two, ..., six". The child with this concept may be directed and learn to utter, say "6--7-8-9-10" to solve 6 + 4, but "10" does not mean a numerical structure (composite unity) (Steffe et al., 1982). "Ten" is taken as the sensory-motor result of counting a collection of items. A child who counts "6--7-8-9-10" with or without the production of unit items (perceptual, motor, verbal) is said to be using "partial counting" (Brownell 1928, p. 74) or a counting-on strategy (Fuson, 1982). Unless there are other behavioral indications that a child can construct a numerical structure (see below), the use of partial counting or counting-on strategy will indicate that the child has a preoperational concept of addition.

Any sensory-motor counting activity that is continued after a pause, by counting motor or verbal unit items is said to be an intuitive extension (Steffe et al., 1982, p. 139). Both partial counting and counting-on, if carried out with the production of motor and verbal unit items will be examples of intuitive extension-with-curtailment. The characteristic strategy used by children with the preoperational concept of addition is called intuitive-extension-without-curtailment. The counting activity in the latter strategy begins with "one" rather than the next number word after the first addend. Brenda, a counter in the motor period from May 1981 to February 1982 (cf. Steffe & Cobb, 1982), provided a behavior that was cahracteristic of the preoperational concept stage. She counted a sequence of her motor acts to solve a task involving 9 screened felt stars and 4 screened felt stars. She also used the same procedure to solve a following task where she was directed to imagine 50 screened stars and 5 screened stars. Before she carried out the counting activity, the interviewer asked her to pretend she had already counted up to 50, and continue counting but she could not do so despite the prompting. She, however, curtailed counting the number word sequence preceding 50 to solve the next task 50 + 6. But she could not curtail counting the sequence for 9 to solve 9 + 4, a task she had solved some few minutes earlier. Brenda remained in the preoperational concept stage when she entered the verbal period in February 1982. To solve 15 + 9 she uttered (without any finger movements) "1-2-...-15", then continued "16-17-...-24" while sychronously extending fingers.

Brenda did not spontaneously curtail the number word sequence for the first addend. Counting a sequence of motor or verbal unit items for the first addend was crucial for her. Curtailment was possible only when the appropriate sequence had already been uttered in the immediate past.

Operational Concept Stage

The ability to solve a missing addend problem is taken as the basic characteristic behavior of children with an operational concept of addition. The additive strategy involves making an intuitive extension-with-curtailment and the child keeping track of the counting activity beyond the first addend. The child who uses the strategy intends to use the records to determine the unspecified numerosity of the missing addend. The operation of uniting distinct unitary items into a composite unity is called integration (Steffe et al., p. 111). The result of an integration is therefore a numerical structure. Scenetra, a counter in the abstract period from March 1981 to June 1982 (cf. Steffe & Cobb, 1982), was presented a missing addend problem involving 11 squares, 6 of which were visible and the rest screened. She uttered "six" (without the preceding number word sequence) then sequentially extended her five left fingers while subuttering number words. When she counted to "11", she paused, looked at her open left hand and said, "There's five under there". The interviewer asked how she arrived at the result, and she replied, "I put six in my head and counted on five more."

Scenetra structured her counting scheme to provide a means for determining the numerosity of the screened squares. She took her utterance, "six" as referring to the counting acts of uttering the number word sequence "one, two, ..., six". This was a tacit integration (Steffe, et al, 1982 p. 112) because it was carried out without words or speech and was implied in Scenetra's solution when she recorded her counts starting with "seven". Scenetra's counting is called operative (Steffe, 1982) because it is an expression of a numerical structure. Her count, "7-8-...-11" is refered to as making a numerical extension (Steffe, et. al, 1982, p. 151) and is an example of Countin-on (Fuson, 1982, p. 69) and also of Adding-on (Davydov and Andronov, 1981). Children who solve addition problems by making a numerical extension are said to be in the operational concept stage.

Arithmetical Concept Stage

A child is said to have an arithmetical concept of addition if the child has habituated simple addition facts and can recall them to solve difficult addition problems (from the child's point of view). For example, Tyrone, a counter in the abstract period from February 1981 to June 1982 (cf. Steffe & Cobb, 1982), quickly responded "28" to 13 + 15 (expressed with felt numerals). When the interviewer asked, "how did you do that so fast?" He replied, "because 13 and 13 is 26 and 2 more will be 28".

Tyrone had not only acquired the capacity to construc abstract number but could also decompose number spontaneously and confidently to solve addition combinations which he could not recall immediately. Tyrone's solution is an example of the use of thinking strategies for finding sums (Cobb, 1982). The thinking strategies are synonymous with Brownell's (1928) methods of grouping and convertion, with Carpenter's (1980) heuristic strategies, Carpenter and Moser's (1982) derived facts, Houlihan and Ginsburg's (1981) indirect memory, and Thaeler's (1981) rational methods. The child who has constructed an abstract number (composite unity) and can in addition use any of the thinking strategies is said to be in the arithmetical concept stage.

CONCLUSION

Counters in the perceptual period remained throughout in the sensorimotor concept stage. Counters in the motor period were observed in the sensorimotor concept stage and all advanced to the preoperational concept stage but progressed no further. Counters in the verbal period were observed in the preoperational concept stage and some advanced to the operational stage but progressed no further. Counters in the abstract concept stage were observed in the operational concept stage and all advanced to the arithmetical concept stage.

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FIRST GRADERS INVENT ARITHMETIC:
PIAGET'S THEORY USED IN THE CLASSROOM

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I hypothesized on the basis of Piaget's theory that addition does not have to be taught, since children construct number on their own, and number itself contains addition. (The number series is constructed with the operation of +1.) The approach I experimented with eliminated instruction and used only two kinds of activities: situations in daily living such as the counting of votes and group games. The experimental group who engaged only in these activities memorized sums slightly better by the end of first grade than a comparison group who had traditional instruction with worksheets. Data are also presented to show that place value is an inappropriate objective for first grade. Since first graders are still constructing number with the operation of +1, it is impossible for them simultaneously to build a second level involving higher-order units of 10.

ADDITION

It has traditionally been assumed that addition has to be taught to first graders. However, I drew the following implications from Piaget's theory:

1. Addition does not have to be taught, since children construct number on their own, and number itself contains addition. (The number series is constructed with the operation of +1.)
2. Feedback from the teacher about the correctness of the answer is unnecessary because, in logico-mathematical knowledge, the ultimate source of feedback is the internal coherence of the system.
3. Social interaction with other children and adults is important for the development of logico-mathematical knowledge, as it stimulates mental activity through the confrontation of viewpoints in a context that obliges the child to be coherent (Piaget, 1947; Inhelder, Sinclair & Bovet, 1974; Perret-Clermont, 1980).

On the basis of this theory, I experimented with an approach to first-grade arithmetic that eliminated instruction and used only two kinds of activities: situations in daily living and group games.

An example of the former is when a class tried to decide whether or not to continue an activity, and 13 children voted in favor of continuing it. One child immediately said, "We don't need to vote for the other choice." The teacher asked, "How do you know that?" and the child explained: "13 + 13 = 26, and we only have 24 kids today." Many children did not understand this reasoning, and the teacher asked if the class wanted to vote for the alternative. Those who did not understand the reasoning were glad to be able to use this empirical method. Those who understood the reasoning lost no time in saying, "We told you 13 was the winner."

An example of a group game is Double War. This game is played just like War, except that the sum of two cards is compared with the sum of the opponent's two cards, and the person who has the greater total picks up all 4 cards. Cards from 2 decks with numbers up to 4 are used at the beginning of the year. They are dealt to 2 players, with 2 piles for each. The children turn the top card of each pile over and compare the sums. The person who has more cards at the end wins.

Table 1 shows the findings from individual tests in which the children responded orally. As can be seen in the column labeled "Experimental group," the children did learn sums without any instruction.

Situations in daily living and games are not new to teachers, but they are usually considered as supplements to instruction. I think these are superior to instruction (with worksheets) for the following reasons:

1. They promote the development of autonomy in children, socially, intellectually, and morally. In games and situations in daily living, decisions are made by children, and children make and enforce rules. They also check each other's answer. In traditional instruction, by contrast, the teacher decides what is to be learned when, and which answers are correct. Children thus learn to be obedient and dependent on the teacher to know the truth, i.e., to be governed by someone else.
2. They make children mentally more active.

   In Double War, children supervise each other constantly and figure out in situations such as the one shown in Figure 1(a) that logical thinking is sometimes better than mechanical addition. They reason that the 2 "2's" are the same, and therefore all they

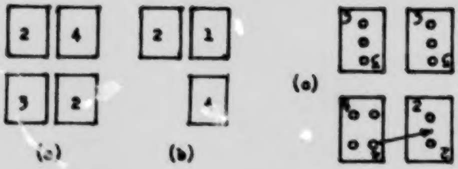
Table 1

Percent of First Graders Who Gave the Correct Answer Immediately

	Experimental group N=24	Comparison group N=12	Difference
2 + 2	100	100	0
5 + 5	100	92	8
3 + 3	100	100	0
4 + 1	100	100	0
6 + 1	100	100	0
5 + 1	100	100	0
1 + 4	100	100	0
2 + 3	100	92	8
3 + 2	100	92	8
4 + 4	96	100	-4
1 + 5	96	100	-4
6 + 6	88	75	13
4 + 2	88	83	5
3 + 2	88	92	-4
2 + 5	88	92	-4
6 + 2	88	83	5
2 + 6	88	67	21
6 + 3	79	67	12
4 + 5	75	92	-17
2 + 4	75	92	-17
5 + 4	71	83	-12
4 + 3	71	83	-12
3 + 4	71	75	-4
4 + 6	67	50	17
5 + 3	63	83	-20
3 + 6	63	67	-3
3 + 5	63	75	-12
6 + 5	54	58	-4
5 + 6	50	58	-8
9 + 1	100	100	0
7 + 2	100	83	17
1 + 10	100	92	8
10 + 10	100	75	25
2 + 8	88	67	21
7 + 3	83	67	16
9 + 2	79	67	12
9 + 9	63	58	5
8 + 5	54	42	12
8 + 8	54	42	12
7 + 7	50	50	0
5 + 7	50	58	-8
7 + 8	38	25	13

have to compare is the "3" and the "4." When they complete worksheets, by contrast, they are encouraged to work in a mechanical way, as they have to produce answers to all the questions.

Figure 1



They also think at times of not turning the 2 cards over simultaneously and create situations such as the one shown in Figure 1(b). In this case, they say that they do not need to see the other card because "4" is already more than 2 + 1. In learning to add numbers, it is desirable to be mentally active and to know when it is more intelligent not to add them.

A child who did not know 3 + 3 and 4 + 2 once said to the teacher immediately, "It's the same thing." When the teacher asked her how she knew so quickly that the sums were the same, the child explained (see Figure 1(c)) that by moving one of the symbols from the "4" card to the "2" card, she would have "3" and "3." This is another example of active thinking.

Finally, children sometimes decide to count the cards that were dealt. First graders cannot always figure out what happened when they find unequal numbers, and some do not even care. This is nevertheless the beginning of division, and active minds start working on division at their level in their own way.

Note that the above evaluation was not based on a comparison of the experimental group with another group. I do not believe that a "method" that produces more memorization by the end of first grade is necessarily a better one. It is nevertheless interesting to compare the experimental and comparison group in Table 1. The following observations can be made:

1. There is no consistent difference between the two groups when small addends up to 6 are involved.
2. The experimental group is consistently superior when larger addends (from 7 to 10) are involved.

The results discussed so far concern answers that were given orally by the children. What about their ability to write answers? The children in the experimental group did their arithmetic without pencils. The decision not to require writing was made because the distinction Piaget made between *abstraction* and *representation* led me to emphasize thinking (abstraction) rather than the production of written answers (representation).

Abstraction (reflective abstraction) refers to the creation and coordination of relationships. When we see a red pen and a blue one, for example, we may think that they are *different*. This difference exists neither *in* one pen nor *in* the other, nor anywhere else in external reality. It is a relationship created by the individual who puts the 2 objects into this relationship. The 2 pens can also be considered *similar*. This is another relationship created by the individual, through reflective abstraction. The individual can also put the pens into the relationship of "*two*." When he puts "two" and "two" into an additive relationship, this is also an example of reflective abstraction.

An individual who has created these relationships can represent them in a variety of ways. The relationship "two," for example, can be represented orally with the word "two," or graphically with "oo" or "2."

This distinction between abstraction and representation led me to hypothesize that once the child has constructed sums (by reflective abstraction), he will be able to represent this knowledge on paper with the greatest of ease. I, therefore, let the children be free to think and talk, without burdening them with having to write answers. When we gave them worksheets as tests, they had no difficulty putting their knowledge of sums on paper.

PLACE VALUE

All the first grade arithmetic programs I know of teach place value (by giving bundles of 10 straws, having children say and write "*2* tens and *5* ones," etc.). However, place value is cognitively impossible for first graders to understand and is, therefore, an inappropriate objective. I base this statement on the following findings and Piaget's theory and research on number.

The method I used was an adaptation of the one devised by Mieko Kamii (1980, 1982). With this technique, she had found the percents shown in Table 2 of children at various age levels who interpreted the "1" in "16" as meaning "10." Here is what I did.

Age	Percent
6	0
7	12
8	17
9	46

Table 2

1. I put 16 chips out and asked the child to count them and to make a drawing of "all these." Almost all the children made drawing like the one shown in Figure 2(a). (Place value had, of course, been taught.)

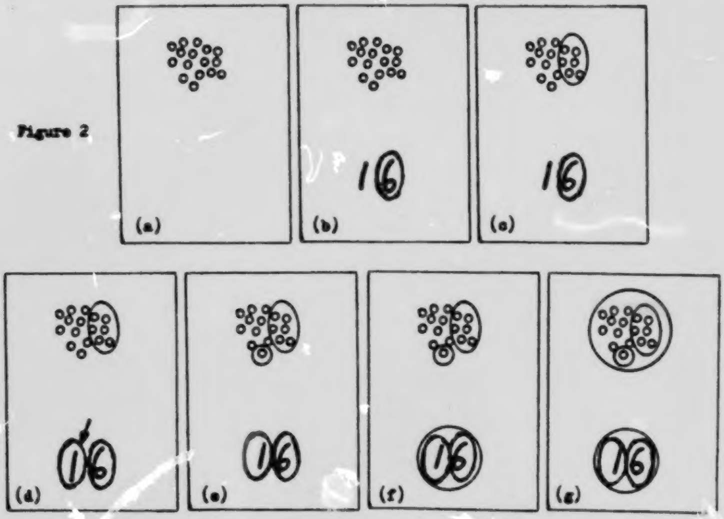
2. I then asked the child to write "sixteen with numbers" on the same sheet, to show that there were 16 chips. No child, even in first grade, had any difficulty writing "16."

3. I then asked the child what "this part" meant, circling the "6" in "16" as shown in Figure 2(b) and asking him to indicate in the drawing what "6" meant. The child usually responded by circling 6 chips (Figure 2(c)).

4. I then asked the child what "this part" meant, circling the "1" in "16" as indicated by the arrow in Figure 2(d). First graders typically circled 1 chip as shown in Figure 2(e).

5. I then asked the child what "the whole thing" meant, circling "16" (Figure 2(f)). All the first graders circled the entire group of objects (Figure 2(g)). When asked why "these" (the 9 left-over chips) were not circled, the child typically explained, "You don't need any circle around those" . . . because the way to write "16" is with a "1" and a "6."

Figure 2



Grade	Percent
1	0
4	51
6	60
8	78

Table 3

Table 3 gives the percents of children at various grade levels who circled 10 chips to show that the "1" in "16" meant "10." The reason for this difficulty is that this task requires the construction of a 2-level mental structure (shown in Figure 3). Since first graders are still building the first level with the operation of +1 (by reflective abstraction), they cannot simultaneously construct the second level involving higher-order units (of 10). For those who do not believe in mental structures, I offer the following additional argument: Place value involves division (26 ÷ 10 = 2, with 6 left over) and multiplication (the "2" in "26" means 2 x 10). These operations are too hard for first graders. They, therefore, cannot assimilate the conventional system of written representation. Empirically making bundles of straws, drawing circles around groups of 10 objects, and writing "2 tens and 6 ones" are one thing. The mental construction of a hierarchical structure is quite another thing (Kamii, 1982).

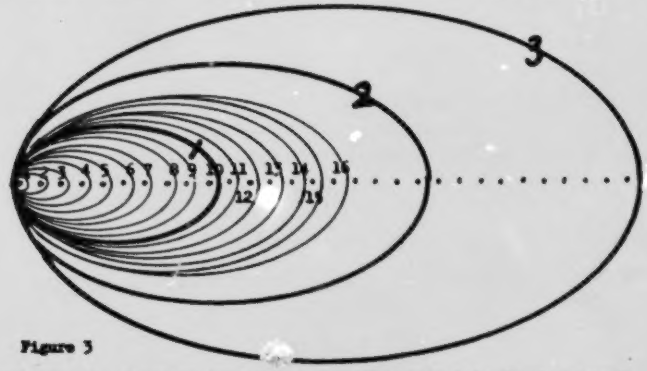


Figure 3

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WHAT ARE THE REFERENTS OF ARITHMETIC SYMBOLS?

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Learning arithmetic includes learning the meaning of symbols. One type of meaning is semantic meaning; a substitution process. Upon presentation of the symbol, one can mentally substitute its referent and think about it as though it were physically present. For example, the symbol 'cat' has semantic meaning when one knows its referent and is able to substitute its referent mentally when confronted with the symbol.

Evidence for semantic meaning in arithmetic is being able to call to mind the appropriate referents when presented with symbols; however, the referents are not obvious. In this paper we extend Van Engen's analysis of meaning which asserts that the referents of arithmetic ultimately reside in overt acts called operations. We propose a nontraditional conception of the role of manipulatives and modes of representation within a broader model of acquiring semantic meaning.

As children learn mathematics we want them to learn the meaning of the symbols. Learning mathematics with meaning is considered essential for any real understanding. But what is meaning and how do we teach so that children can acquire it?

SEMANTIC MEANING

Van Engen (1949) provides a basis for discussing meaning and determining the referents for our symbols in arithmetic. He writes:

> "In any meaningful situation there are always three elements. (1) There is an event, an object, or an action. In general terms a referent. (2) There is a symbol for the referent. (3) There is an individual to interpret the symbol as somehow referring to the referent." p.323.

From this analysis, it is clear that the symbol should not be confused with its referent. Just as "cat" is not synonomous with the four-legged furry animal, so the symbol "$\frac{1}{2}$" should not be confused with the referent it symbolizes. "It is important to remember that a symbol always refers to something outside itself" (Van Engen 1949, p.323). Thus, for arithmetic to be meaningful we must be aware of the referents so that we can substitute them for their symbols in our minds.

What are the referents to which arithmetic symbols refer? Van Engen asserts that "arithmetic is primarily concerned with operations and transformations." (1949, p.324). Arithmetic symbols refer to overt acts and actions such as "grouping of objects in certain ways, to dividing objects in certain ways, to comparing objects in certain ways." (1949, p.324). Van Engen goes on to explain:

> "...It is of utmost importance to note that the word 'operation' is not used in the sense of 'the fundamental operations of arithmetic.' 'Operations,' as used in this paper, designates the referent of a symbol - a written word, a mark on the board, a gesture, or a spoken word. This referent is an action considered with reference to the thing acted upon. Thus, the term 'operations' refers to overt acts. For example, the act of breaking a stick into halves is an operation." (1949, p.326).

It becomes clear that symbols are the products and the written records that communicate the actions taken by the individual who understands the symbols.

OPERATIONAL ARITHMETIC

From the idea that operations are the referents to which arithmetic symbols refer emerges the concept of operational arithmetic. Operational arithmetic is the operations - overt acts and mental images - acquired through direct experience with manipulation of objects that become encoded in symbolic arithmetic. A final quote from Van Engen highlights the importance that must be accorded the concept of operational arithmetic.

> "Considered from this point of view, the whole object of arithmetic instruction clearly is to help the child devise a system of symbols which, in some sense, is representative of a realm of events - a series of symbolized operations with which the child has had direct experience. Then operations, symbolized by the spoken word, or, in the case of mathematics, the mathematical symbols, are the primary instruments of knowledge. *Awareness of the operation or of an event itself is not knowledge. Neither is awareness of the symbol knowledge. When, however, the two becomes associated in the mind of the individual, then there is knowledge (italics mine.) (1949, p.325).*

In summary, an individual knows arithmetic when they can connect symbols (written records) to their corresponding operations (referents) and can reason on operations even as they work with symbols.

An example of the failure to make the connection between symbolic arithmetic and operational arithmetic seems apparent in a clinical interview by Davis (1980) with a student, Marcia. When presented symbolically with subtractions that involved bridging zeros, Marcia responded as shown in Figure 1.

$$\begin{array}{r} 7005 \\ -\ 25 \\ \hline 7070 \end{array} \qquad\qquad \begin{array}{r} \overset{6}{\cancel{7}}\;0\;0\;{}^{1}2 \\ -\ 7 \\ \hline 6\;0\;0\;5 \end{array}$$

Figure 1: Marcia's Subtraction Errors.

Marcia presisted in this error even though she could apply the subtraction algorithm correctly in regrouping situations that did not require bridging zeros. Davis relates that Marcia's teacher "corrected" this error by teaching her a "trick" which she applied correctly to the first example in Figure 2 but also applied to the second example in Figure 2. Marcia is manipulating symbols without meaning.

$$\begin{array}{r} \overset{6}{\cancel{7}}\;\overset{9}{\cancel{0}}\;\overset{9}{\cancel{0}}\;2 \\ -\ 2\;8 \\ \hline 6\;9\;7\;4 \end{array} \qquad\qquad \begin{array}{r} \overset{6}{\cancel{7}}\;\overset{9}{\cancel{0}}\;\overset{9}{\cancel{0}}\;0 \\ -\ 2\;0 \\ \hline 6\;9\;7\;0 \end{array}$$

Figure 2: Marcia's Errors After Reteaching.

Later in the interview Davis finds that Marcia is quite familiar with Dienes' MAB blocks and using them she demonstrates a good deal of facility with place value. She can use the blocks to represent 7002 and to show that 70 can be seven tens or six tens and ten ones. However, there is no evidence that Marcia can carry out the subtraction algorithm in terms of operational arithmetic. Therefore, the connection between a series of actions and the written record of them is not demonstrated. Thus, Marcia has not shown she has semantic meaning for the subtraction algorithm (Kolb, in press). Marcia's behaviors provide an example of a case where facility with manipulatives does not imply semantic meaning.

SEMANTIC MEANING AND MODES OF REPRESENTATION

Bruner has emphasized the importance of human capacity to represent our world in different modes (1967). It has become commonly accepted that the usual course of development of ideas is through an enactive mode followed by an iconic mode followed by a symbolic mode (Post, 1981 p.123). Therefore, the common wisdom of our profession suggests that this progression is the optimum instructional sequence for acquiring meaning and understanding of arithmetic. Figure 3 portrays the accepted relationship among these three modes. An unfortunate and mistaken side effect of this conception of meaningful instruction is the tendency to consider enactive and iconic representations as the referents of symbols.

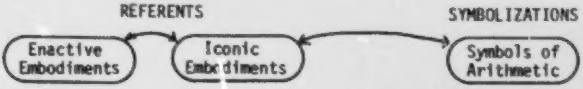

Figure 3: Enactive and iconic embodiments as referents of arithmetic symbols.

Manipulative materials and pictures of objects are not the referents of symbols but are themselves just other symbols for the real referents. We act like manipulatives such as blocks or pictures of "pies" are the referents of symbols. A yellow hexagon with one sixth of it covered with a green triangle is not the referent for 1/6. It is a representation for 1/6 and is just as much a symbol as the symbol "1/6." The picture is not a referent for the symbol 1/3 but is simply another symbol. The referent for 1/3 is the operation of selecting a unit, sharing it with 3 and covering one of the shares (Peck and Jencks, 1981). The fraction symbol is an instruction in coded form for carrying out a sequence of operations. Since the referents of arithmetic symbols are operations and not manipulative materials or pictorial representations, then enactive and iconic representations may not be a required part of a meaningful instructional sequence. The connection to be established is between operations and symbols and not between objects or pictures and symbols. These embodiments may provide the physical stuff upon which the operations can be carried out, however, many students seem to learn the operations and to relate them meaningfully to their corresponding symbols without resorting to an enactive or iconic stage.

In Figure 4 we propose a model that is consistent with the assertion that operations are the referents of arithmetic symbols. Enactive and iconic embodiments as in manipulative materials and pictorial representations are themselves just facilitative symbolizations of the operations.

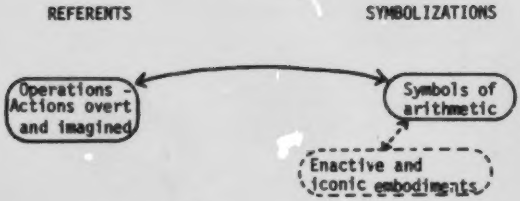
REFERENTS
SYMBOLIZATIONS
Operations - Actions overt and imagined
Symbols of arithmetic
Enactive and iconic embodiments

Figure 4: Enactive and iconic embodiments as facilitative symbolization

Peck and Jencks (1981) reported the results of interviews with above average ninth-graders in which the students were asked for the meaning of various fraction symbols. When asked to show 1/3 one student provided the pictorial representation shown in Figure 5. This was given as an example from a child who does not know the fraction 1/3.

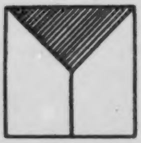

Figure 5: Child's picture of 1/3.

Since the three parts are not equal, the shaded portion to the whole is not representative of 1/3. If we slip into thinking the subdivided and shaded square is the referent for 1/3 then we would say this child does not know the referent for the symbol and that the child lacks semantic meaning.

However in the conception we propose we must think of Figure 5 as a symbol that itself refers to an operation. We cannot judge the child's operations by simply looking at the product of these operations. If the child in making the picture meant to symbolize the operation of selecting a unit, subdividing it into three equal shares and consider one of the three shares, then this child has the referent for 1/3 and hence has semantic meaning. The drawn figure may simply be an imperfectly drawn symbol for a correct thought. As suggested by this alternative explanation of the child's picture and by Figure 4, the true referents of arithmetic symbols are operations. Thus, we cannot judge the existence of semantic meaning on the basis of enactive or iconic symbolizations, but rather must judge it on the basis of knowledge of the operations.

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NUMBER LINE REPRESENTATIONS OF FRACTIONS[1]

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The study explored students' interpretations of representations of fractions on number lines and the effect of instruction on those interpretations. Subjects were five fourth-graders, and instruction was a four-day unit on the use of number lines. A 16-item, multiple-choice pre- and posttest was used along with videotaped interviews. Performance improved except when students had to associate a reduced fraction symbol with an equivalent, unreduced fraction representation on a number line. The difficulty at "unpartitioning" a representation suggests that care needs to be given to developing a concept of equivalent fractions. Further, translations among various models of fractions might foster better performance.

This study (a) explored ways that students might interpret, or misinterpret, the representation of fractions on number lines and (b) determined possible influences of instruction on those interpretations. The number line, which embodies the measure subconstruct of rational numbers (Kieren, 1976), was chosen for study because it is a pervasive model of number representations throughout school mathematics instruction.

As a model for representing fractions, the number line differs from other models; e.g., sets, regions; in several important ways. First, a length represents the unit, but more important the measure construct suggests both iteration of the unit as well as simultaneous subdivisions of all iterated units. That is, the number line can conceptually be treated as a ruler. For example, if one wants to measure with a ruler to the nearest one-seventh of a unit, all units would be subdivided into sevenths. Second, on a number line there is no visual separation between consecutive units. That is, the model is totally continuous. Both sets and regions as models possess visual discreteness. When regions are used, for example, space is typically left between copies of the unit.

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Third, the number line requires the use of symbols to convey part of the intended meaning. For example, point A in Figure 1a has no numerical meaning until at least two reference points are labeled. Two possible meanings are given in Figures 1b and 1c. Figures 1d and 1e, however, do convey meaning without any accompanying symbols, though their interpretation requires some standard conventions about the nature of a unit.

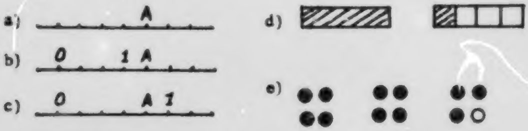

Figure 1. Representations of Fractions

The use of symbols to label points on a number line may focus a student's attention on those symbols rather than on the pictorial embodiment of the fractions. This focusing may in turn cue symbolic processes as the predominate mode of manipulation of information. Too, the necessary but not directly used marks on a number line may act as perceptual distractors (Behr, Post, Lesh, & Silver, 1982).

These differences between the number line and other models suggest that students may interpret number line representations only with some difficulty. Too, instruction may have to be carefully keyed to the critical attributes of number line representations in order to be effective.

METHODS

Subjects

Subjects were five fourth-graders in an elementary school in northern Illinois; three boys and two girls. They were selected, through teacher evaluations, to represent a cross-section of facility with arithmetic concepts and were also subjects in an 18-week teaching experiment (Behr, Lesh, & Post, Note 1).

Instruction

Instruction was a four-day lesson concerning association of fraction concepts, relations, and operations with points, comparisons, and transformations on a number line. Specific objectives were as follows:

1) To associate whole numbers, fractions, and mixed numbers with points on the number line.
2) To use the number line to help connect "improper" fraction names to "mixed number" names.
3) To use number lines to determine which of two fractions is less or whether they are equivalent.
4) To use number lines to generate equivalent fractions.

The lesson on number line representations was presented near the end of the larger teaching experiment. The fraction test of Novillis (1980) was given immediately prior to and immediately after the instruction.

Test

Novillis's 16-item, multiple choice test can be partitioned into two 8-item subscales in several ways: (a) fraction given with representation to be chosen versus representation given with fraction to be chosen, (b) number line shows 0 to 1 versus number line shows 0 to 2, and (c) representation on number line shows unreduced fraction versus representation shows reduced fraction. For each item there were five choices, one of which was "Not Given"; this choice was never the correct choice. In all cases, the fraction symbol in the correct fraction/representation pair was reduced even if the representation was for an unreduced equivalent fraction.

RESULTS

Scores on the six various possible subscales are given in Table 1. For five of the six subscales, performance uniformly increased or remained constant from pretest to posttest. The sole exception was when the representation was unreduced and the fraction symbol was reduced. As a follow-up of this subscale, scores were separated according to the other categories of items (Table 2). With the exception of Student 1, students were unable to choose a reduced fraction name when an unreduced equivalent form was represented on a number line.

To help determine what processes the students might be using, incorrect responses on the unreduced representation subscale were examined. On

Table 1: Subtest Scores[a]

Subtest	Student					Number of Times Posttest-Pretest		
	1	2	3	4	5	>0	=0	<0
fraction given	4(8)	4(4)	2(4)	1(4)	2(4)	4	1	0
representation given	4(8)	2(4)	2(2)	1(3)	0(3)	4	1	0
number line 0-1	8(8)	5(5)	2(3)	2(4)	1(4)	3	2	0
number line 0-2	0(8)	1(3)	2(3)	0(3)	1(3)	5	0	0
reduced representation	4(8)	5(7)	3(6)	1(6)	0(7)	5	0	0
unreduced representation	4(8)	1(1)	1(0)	1(1)	2(0)	1	2	2

[a] x(y) means x = pretest score, y = posttest score. Each subtest had a maximum possible score of 8.

Table 2: Refined Scores for Unreduced Representation Subscale[a]

Subcategory	Student				
	1	2	3	4	5
fraction given with					
0 to 1 number line	2(2)	1(1)	1(0)	1(0)	1(0)
0 to 2 number line	0(2)	0(0)	0(0)	0(1)	1(0)
representation given with					
0 to 1 number line	2(2)	0(0)	0(0)	0(0)	0(0)
0 to 2 number line	0(2)	0(0)	0(0)	0(0)	0(0)

[a] x(y) means x = pretest score, y = posttest score. Each subcategory had a maximum possible score of 2.

the pretest, 10 of the 31 incorrect responses were "Not Given"; two were blanks. On the posttest, however, 28 of the 30 incorrect responses were "Not Given"; none were blanks.

Additional information was available from videotaped interviews. In three interview tasks, students were to find equivalent fractions, $\frac{5}{3} = \frac{\square}{12}$, $\frac{8}{6} = \frac{\square}{3}$, and $\frac{8}{6} = \frac{\square}{12}$. Student 1 solved these problems symbolically and used the number line only to verify the solutions. Student 4 used counting strategies but solved all three correctly. Student 2 combined number line and symbolic algorithms and solved only the first and third tasks involving larger denominators. Students 3 and 5 used addition and subtraction strategies and solved the last two tasks correctly, but possibly only because of the 2:1 ratio of the demoninators.

DISCUSSION

In many ways the instruction seems to have been quite effective. Except for Student 1, there was no improvement when the number line representation was of an unreduced, but equivalent, fraction. In particular, when the representation was given and the reduced fraction was to be selected, only Student 1 gave any correct responses.

Only Student 1 spontaneously used symbolic algorithms to generate equivalent fractions in the interview tasks. Clear access to those algorithms may indicate a well developed concept of equivalent fractions which allows spontaneous reducing of fractions as well as recognition of unreduced fractions. Students who did not have access to those algorithms may not have recognized when a fraction was not reduced.

In the instruction on equivalent fractions two approaches were used: (a) given two fraction symbols, determine if they name equivalent fractions and (b) use a set of number lines already divided into halves, thirds, fourths, etc. (essentially, they were fraction bars) to write fractions equivalent to a given fraction. There were no examples of writing a fraction symbol for a given representation and then rewriting the symbol in an equivalent form, though there were some examples of changing the representation to match a given fraction; e.g., drawing a representation for $\frac{3}{4}$ and then changing the representation to $\frac{6}{8}$. The students apparently did not see the reversibility of the associations and were not able to transfer the process.

OBSERVATIONS AND IMPLICATIONS

Students difficulty at adding partitioning points to generate higher term fractions or mentally "removing" partitioning points to generate lower term fractions is not unique to the number line model (Behr, Post, Lesh, & Silver, 1982; Payne, 1976). Moreover, the greater difficulty children have generating lower term fractions by "unpartitioning" pervades children's dealings with both continuous and discrete models.

A similar phenomenon is observed with symbolic equivalent-fractions-tasks: Generating higher-term fractions seems to be easier for the children than generating lower-term fractions. At the symbolic level, this difference in difficulty may be due to children's greater facility with multiplication than division.

With manipulative-aid tasks, however, the children seem to rely heavily on the visual representation of a fraction; flexibility in the perception of equivalent fractions independent of the given representations is not yet achieved. Children not only seem distracted by "extra" lines, but also seem to question "the rules of the game." That is, some children have been observed to add partitioning lines but when faced with the "removal" of lines these same children hesitate and may query the teacher or interviewer about whether it is "alright to take out lines." Other children, however, have been found totally unable to perceive lower-term fractions in the presence of extra lines. More generally, this partitioning/unpartitioning phenomenon seems to pervade many children's work with most models for rational numbers.

A major hypothesis of the research project, of which this study is one part, is that it is translations between and within modes of representation which facilitates learning (Behr, Post, & Lesh, Note 1). As noted earlier the instruction provided models of translations of three types: (a) symbol → number line, (b) symbol → number line → number line, and (c) number line → symbol → symbol. Inclusion of translations such as symbol → number line → number line → symbol might have helped children make symbol → symbol translations in generating equivalent fractions (see Figure 2).

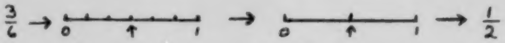

FIGURE 2.

Translations between different kinds of models might also have been helpful. That is, children might have used a region model to generate equivalent fractions; e.g., $\frac{2}{3} = \frac{\square}{9}$; and then translated their work to the number line to accomplish a comparable solution. Seeing the relationship between models might have facilitated their ability to use the number line.

Finally, knowledge of equivalent fractions seems to be important to the full utilization of number line representations. Knowledge that is developed only through symbolic algorithms may be isolated and not called upon in the context of manipulative tasks. More reasonably, work with the number line during instruction on equivalent fractions would probably be called for. For example, partitioning units of a number line first into halves, then fourths, etc., would illustrate the notion that to every point on the line there is associated many equivalent fractions. Before using the number line (e.g., to model addition and subtraction, especially of unlike fractions) more skill with equivalent fractions in the context of the number line is essential. Automatic generation of equivalent fraction representations, through further partitioning or unpartitioning of the number line "in the mind's eye", could facilitate flexibility in perception. Such flexibility seems to enhance students' performance.

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DEVELOPMENT OF FRACTIONAL NUMBER CONCEPTS FOR ADOLESCENT LEARNERS

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Noelting's research of cognitive development of proportional reasoning in different fractional contexts was replicated and extended by analyzing data from a conceptual point of view. Results substantiated Noelting's stage analysis of the development of proportional reasoning in the ratio subconstruct and also supported his findings that symbol manipulations are not connected to related problem solving strategies. But differences were found between the conceptual development of ratio and division subcontructs by analyzing subjects' problem solving strategies from a conceptual point of view rather than looking at the development of proportional reasoning in ratio and division contexts.

Fractions embody coordination of two variables in ratio, division, operator, and part-whole conceptual forms or subconstructs (Kieren, 1976, 1980). Noelting (1980a, 1980b) has identified two major periods and several stages in the development of proportional reasoning along the dimensions of item-structure and problem solving strategies within the ratio subconstruct (the Orange Juice Experiment). Noelting (Note 1) found the same periods, stages and problem solving strategies in a study of proportional reasoning using the problem solving context of sharing cookies within the division subconstruct (the Sharing Cookies Experiment).

This study is basically a replication of Noelting's and Gagne's study (N&G, 1980), but differs in that the data was analyzed from both Noelting's development of proportional reasoning point of view and from a conceptual point of view that was designed to account for differences among subconstructs by distinguishing among states of knowledge as they relate to concepts (abstractions), concrete experiences associated with concept functioning, and symbol manipulations. The major purposes of the present study (M&S) are to substantiate Noelting's results with ratios for English speaking subjects, and to search for differences in the conceptual development of the ratio and division subconstructs, making as fine of distinctions as possible by concentrating on conceptual development of subconstructs rather than on the the development of proportional reasoning in different contexts.

METHOD

Three instruments were given to five intact classes, one each from grades five through nine in a principally rural school division. The orange juice instrument was given first, and then the numerical fractions instrument was given during the same class period. About one week later, the sharing candy instrument was given during part of a class period. Each class was selected to represent an average ability range of students. No classes were receiving fraction instruction during the time of the testing.

All three instruments consisted of the same 12 items, though they were presented in their respective experimental formats: the orange juice instrument had rows of glasses filled with orange juice sketched over a row of glasses filled with water; the sharing candy instrument had rows of candy bars (instead of cookies as used in N&G) sketched over rows of people and the numerical fraction instrument had numerators sketched over denominators in regular fractional form. All items were presented in the form of comparative tasks: which situation has more orange taste? which group of people gets more candy? which fraction is larger? Subjects were asked to choose whether the left side was more than the right side (answer A), whether the two sides were equal (answer E), or whether the right side was more than the left side (answer B). At Noelting's suggestion (Note 2), item <1/1 vs. 2/2>was substituted for item <6/2 vs. 8/3> used in the N&G study. Items one through seven were corrected as passed or failed according to the subjects' choice; items eight through 12 were corrected in accordance with both the choice and explanation given (the explanation had to show evidence of an appropriate strategy). Each instrument corresponded to its respective experiment.

A scalogram analysis was performed for each experiment using Noelting's adaptation of the Guttman procedure where all failures at items statistically easier than the last item passed were considered as errors in the scalogram. The CR coefficient of reproducibility and PPR coefficient of scalability were calculated using the Jackson computational formulas (White, 1957) for each experiment. A CR value of 0.90 (about 10% errors) represents an acceptable value and a PPR value of 0.70 is considered satisfactory.

Strategies were classified according to the (i) subconstruct in which they were applied, (ii) whether they were applied to concrete states or symbol states, (iii) whether they were appropriate or inappropriate, (iv) successful or unsuccessful, and (v) according to further individual details unique to each identifiable strategy. For example, consider this hypothetical response to item 10, <2/3 vs. 7/9>, as it occurs on the sharing candy instrument: "On the left side three people get two candy bars. On the right side each group of three people get two candy bars and there is one candy bar left. Therefore, B has more candy." This type of response would be classified as an appropriate, successful strategy applied to a concrete state in the division subconstruct. It differs in detail from a similarly classified strategy that completely distributes the candy, since in this case it is not necessary to actually determine the exact amount of candy each person received to decide which side gets the most candy because of the nature of the comparative task.

RESULTS AND DISCUSSIONS

The scalogram analysis for the orange juice experiment in this study correlated almost perfectly with N&G results; the heirarchies were virtually the same -- even the CR's were the same, though M&S's PPR value of 0.56 was relatively low compared to N&G's 0.72 (see Table 1 for a comparison of the studies' hierarchies). Notice that in both N&G's and M&S's sharing cookies (candy) experiments, subjects did better with the improper form items <3/1 vs. 5/2> and <6/3 vs. 7/4> than with the proper form items <3/5 vs. 6/10> and <4/6 vs. 6/9>. This reversal of item frequencies between Noelting's Piagetian labeled stages IIB and IIIA in both studies is evidence in support of signigicant differences between ratio and division subconstructs.

Before looking at the conceptual evidence to support differences between the ratio and division subconstructs, a more detailed discussion of theoretical differences will help in making finer distinctions. Ratio is a correspondence between sets, whereas division is an action applied to a given quantity. Thinking of concepts as equivalence classes, a ratio is the class of all correspondences equivalent to any particular correspondence in the class. Determining whether a ratio belongs or does not belong to a class requires proportional reasoning; therefore it follows that acquiring proportional reasoning is the same as acquiring the concept of ratio. Division can also be thought of as being an equivalence class -- a class of actions -- all performing the equivalent equipartitioning scheme. For example, two actions belong to the same class if and only if their resulting partitioned portions have the same quantity (i.e., their quotients are equal). So in summing up, the ratio and division subconstructs are quite distinct from a theoretical point of view; proportional reasoning should dominate subjects' responses on the orange juice instrument, and divisions and comparing quotients should dominate subjects' responses to items on the sharing candy experiment.

The conceptual evidence supports this theoretical position (see Table 2). First of all, notice that subjects were more successful with candy items than orange juice items; the data supports the relatively late acquisition of proportional reasoning compared to the acquisition of division concepts and manipulative skills with symbols. Secondly, there were only five subjects who partitioned orange juice within the state of concrete experiences compared with 35 who partitioned candy items. And finally, notice how many more subjects operated within the symbol state for the sharing candy experiment compared with the symbolic activity for the orange juice experiment.

Table I

Comparison Between Studies of the Frequency of Success of Items in the Orange Juice, Sharing "Candy," and Numerical Fraction Experiments

Item	Composition	Stage	O.Juice		S. Candy		H. Fract.	
			M&G	M&S	M&G	M&S	M&G	M&S
1	2/5 vs. 4/5	IA	99*	98	98	99	98	97
2	1/2 vs. 1/3	IB	100	98	100	100	93	83
4	1/3 vs. 3/9	IIA	77	38	86	60	85	71
5	2/4 vs. 3/6	IIA	79	59	91	73	89	71
6	3/5 vs. 6/10	IIB	68	33	69	41	85	71
7	4/6 vs. 6/9	IIB	48	28	43	33	74	60
8	3/1 vs. 5/2	IIIA	46	16	73	53	79	45
9	2/3 vs. 7/9	IIIA	41	14	48	22	69	33
10	6/3 vs. 7/4	IIIA	45	19	75	44	85	52
11	3/5 vs. 5/8	IIIB	21	3	19	18	58	33
12	8/5 vs. 5/3	IIIB	21	5	18	18	54	28

*Frequencies are given in percentages.

Table 2

Frequency of Items 8 - 12 Responses by Categories
for the Orange Juice and Sharing
Candy Experiments

	Concrete Experiences							Symbols					
	Inappropriate				Unsuc	Successful				Unsuc		Inapp	
G[a]	Not[b]	Cen[c]	Qua[d]	Add[e]		Par[f]	Who[g]	Div[h]	Com[i]	Div	Com	Per[j]	Fra[k]
						Orange-Juice-Experiment							
5	11	40	40	59	6	1	3	2					2
6	5	45	46	14			2	4	7	3	1		3
7	6	23	37	30	5		7	3	5		2		22
8	10	13	39	37	4		13		4		1		5
9	4	4	14	26	18	4	13	10	2	3		4	17
						Sharing-Candy-Experiment							
5	34	39	36	33	14	2	6	6	4		1		
6	4	36	24	20	14	5	8	8	4	1			16
7	8	6	13	3	15	7	14	9	10	15	1		33
8		20	35	6	3	9	12	2	12				9
9	6	7	2	6	16	12	20	11	14	5	3		13

[a] "G" indicates grade level of participating classes.

[b] "Not" indicates no identifiable strategy.

[c] "Cen" indicates a strategy that centrates on one variable.

[d] "Qua" indicates a strategy that took into account both variables but was either inappropriate or without substance.

[e] "Add" indicates additive strategies.

[f] "Par" indicates a strategy that parititions the candy completely.

[g] "Who" indicates a strategy that compares situation with whole candy bars or glasses without partitioning.

[h] "Div" indicates division with symbols.

[i] "Com" indicates the common denominator strategy.

[j] "Per" indicates the use of percents.

[k] "Fra" indicates a strategy with symbols in fractional form that showed no particular method, but instead was mostly descriptive.

In conclusion, the empirical evidence supports the theoretical position that there are significant differences between the ratio and division subconstructs -- differences in age of acquisition, in stage of differentiations and in problem solving strategies used to make comparisons.

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REASONING WITH UNKNOWNS IN GRADES FOUR, SIX, AND EIGHT

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This study was designed to investigate the development of reasoning with unknowns between grades four and eight. Ninety-six students in grades four, six, and eight participated as subjects. It was found that major differences in performance and error type emerged between the fourth and sixth grade only. The fourth graders were less likely to work backwards (use the inverse of the mathematical operations given in the problem to reach the solution) or succeed on two-step problems. Their errors suggest that they failed on simple problems by not applying, or possessing, the correct strategy and they also failed more complex problems by not attending to all the relevant information present.

Word problems have a reputation for presenting great difficulties to many students (e.g., Goldin & McClintock, 1979). Consider the following example (Goldin & McClintock, 1979): "Phil has seven more than three times as many toys as he had last year, and he has 31 toys. How many toys did he have last year?" The reasoning pattern at the core of this problem requires the solver to operate upon the unknown numbers of Phil's toys at present and last year. We have therefore introduced the phrase, "reasoning with unknowns" (Karplus, Pulos & Stage, 1981) to identify this reasoning pattern.

It has been recognized that the difficulty of word problems depends on the placement of the missing value (e.g., Riley, Greeno & Heller, in press). Problems in which the missing value (N) is to be found from data given earlier (e.g., 5 + 6 = N) are easier than problems in which the starting set has the missing value (e.g., N + 5 = 11). Problems in which the missing value is in the starting set require the person to conceive of an unknown which is operated upon while problems in which the missing value is in the solution do not.

Research suggests that children can successfully reason with unknowns with simple one-step problems (e.g., $N - 3 = 4$) (Riley et al., in press), but not with two-step problems (e.g., $(N-3) \times 4 = 8$) (Goldin & Caldwell, 1979; Karplus et al., 1981). There is substantial progress in solving two-step unknown problems between elementary and middle school. However, the nature of the change in reasoning that occurs is not known. It is the purpose of this paper to investigate this change. Tasks were designed to answer the following questions:

1. What forms of reasoning do children and young adolescents apply on selected word problems?
2. How is reasoning affected by the complexity of the problem and by the subjects' gender and grade level?

METHOD

Subjects

The subjects were 96 students in the fourth, sixth, and eighth grades (16 males and 16 females in each) from a middle class, ethnically mixed school district, performing near the average on state-administered achievement tests.

Material and Procedure

Reasoning with unknowns was investigated with the Number Puzzle Task (Karplus et al., 1981). This task consists of three one-step problems, three two-step problems, and two loop puzzles, shown in Table 1. The interviewer presented the puzzles as follows: "I am thinking of a number. I add 12 to my number and then multiply by 6. I get 90. What is my number?"

Table 1
Number Puzzles in Equation Form

Puzzle	Equation	Puzzle	Equation	Puzzle	Equation
1	Nx5=65	4	(N+12)x6=90	7	(Nx4)-21=N
2	N-12=9	5	(Nx4)-12=32	8	(N-16)x3=N
3	N÷8=2	6	(N÷12)+11=16		

Each puzzle was typed on a card which was placed before the student. It was read aloud by the interviewer, who pointed to the individual steps as they were read. Paper and pencil were furnished for the students and the interviewer offered to carry out any arithmetic operations requested. If the student arrived at an answer, he/she was asked, "How did you figure it out?"

RESULTS AND CONCLUSIONS

The means and standard deviations of the proportion of items solved within each problem type are presented in Table 2. A grade X gender X problem type ANOVA on these scores, with repeated measures on the problem type, found significant main effects only for grade (df 2,90; F=51.39, $p < .001$) and problem type (df 2,180; F=18.65, $p < .001$). None of the interactions was significant beyond the .05 level. Post-hoc comparisons with Newman-Keuls tests at the .05 level found that fourth graders performed significantly less well than sixth and eighth graders, who did not differ amongst themselves. Post-hoc comparisons for problem type found one-step problems easier than two-step and two-step easier than loop puzzles.

Table 2
Means and Standard Deviations of the Proportion of Items Solved Within Each Problem Type

Grade	One-Step Problems		Two-Step Problems		Loop Problems	
	X	SD	X	SD	X	SD
4	.29	.37	.16	.31	.16	.30
6	.79	.30	.79	.35	.63	.38
8	.92	.19	.85	.25	.64	.39

Error Analysis

To understand the nature of these differences better, the errors committed and strategies used were examined. The proportion of subjects at each grade are presented in Table 3, which shows that about half of the errors in each problem were committed by students in the fourth grade.

The following categories were used to classify the errors: 1) Inappropriate Procedures (IP), where students guessed without checking their answer or simply stated that they did not know how to solve the problem; 2) Direct Computation (DC), where students applied arithmetic operations to the given numbers without regard to the structure of the problem, usually by treating the unknown as the result of the operation, e.g., in problem 2, giving an answer of 3; 3) Partial Encoding (PE), where the student used only part of the information in the problem, i.e., in two-step problems they neglect some of the numbers and/or operations; 4) Application Errors (AE), where subjects carried out correct procedures but committed computational errors or where the subject correctly works forward but gives up before he or she reaches the correct solution.

Table 3
The Proportion of Error Responses of Each Type
Committed in Each Problem at Each Grade

Error Type	One-Step Problems			Two-Step Problems			Loop Problems		
	Grade			Grade			Grade		
	4	6	8	4	6	8	4	6	8
IP	.15	.00	.00	.35	.04	.07	.40	.17	.14
DC	.73	.85	.50	.11	.17	.07	.15	.00	.09
PE	.00	.00	.00	.42	.24	.00	.17	.00	.09
AE	.12	.15	.50	.12	.55	.86	.28	.83	.68
N	59	20	10	81	29	14	82	36	35

Examining the errors across problems, it appears that subjects fail the one-step problems when thev either do not have or else fail to use an appropriate strategy, while in the two-step problems there are subjects who also fail to carry out the strategies successfully. Across grades, the errors committed by the students in the fourth grade reflect major problems in activating and coordinating the correct strategy, while those of the eighth graders reflect minor errors in applying the strategies. The errors made by the sixth graders are intermediate.

Strategies

Three successful strategies were identified -- working forward, working backwards, and using equations. Table 4 presents the percent of students who used each strategy at least once, regardless of simple application errors, in each grade on each puzzle. In the one-step problems a shift with age occurs, from primarily working forward in the fourth grade to primarily working backwards in the eighth grade. In contrast, on the two-step problems, the number of students working backwards is larger than the number working forward in all grades, but the proportion is small in grade four and increases across grade. Thus, most of the students who can do two-step problems can work backwards. On the loop puzzle a major shift occurs between grade six and eight, as about a quarter of the eighth graders used equations and none of the sixth graders did so. Equations are seldom used until difficult problems are encountered.

Table 4
The Proportion of Correct Strategies
of Each Type in Each Problem at Each Grade

Strategy	One-Step Problems			Two-Step Problems			Loop Problems		
	Grade			Grade			Grade		
	4	6	8	4	6	8	4	6	8
Equations	.00	.00	.00	.00	.00	.05	.00	.00	.24
Backwards	.42	.60	.79	.48	.51	.51	--	--	--
Forward	.58	.40	.21	.52	.49	.44	1.00	1.00	.76
N	43	80	91	25	83	94	34	87	84

In summary, the research suggests that major changes in the ability to reason with unknowns occurs between the fourth grade and sixth grade with relatively little change between the sixth and eighth grade. It appears that the change entails major acquisitions in both strategies and the executive structures for coordinating the execution of the strategies. In order to further understand the development of reasoning with unknowns, future research should focus on teaching children the prerequisite schemes and examine the cognitive correlates of the acquisition of these skills.

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JUST HOW IMPORTANT ARE LANGUAGE SKILLS IN MATHEMATICAL PROBLEM SOLVING?*

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There has been an increasing interest over the last few years in questions concerning the role that language plays in the problem solving process. For bilingual students who may not have the same facility with their second language as native speakers, the question of how language proficiency affects problem solving performance takes on particular meaning. This presentation will review several research studies where we have attempted to compare and contrast the performance of bilingual Hispanic students with that of monolingual Anglo students in various problem solving tasks covering algebra and logic. Results indicate that language skills play an important role in the problem solving process for both groups, but more so for the Hispanic group.

For the past three years, we have been attempting to identify any specifically language-related difficulties which might hinder the academic progress of bilingual Hispanic college students enrolled in science-related programs. Our investigations have focused on the problem solving skills of these students as this is an indispensable skill for anyone pursuing or practicing a technical profession. In our effort to isolate problems unique to Hispanics, however, we have found that mainstream students often exhibit difficulties similar to those exhibited by Hispanics. As the results of this presentation will indicate, there is a need for greater integration in the development of verbal and mathematical skills for all technical college students.

Unless otherwise stated, the results reported in this paper come from studying the performance of 60 bilingual Hispanic students and a comparison group of 73 nonminority Anglo students. All of these students were enrolled in technical programs with the majority majoring in engineering. The remainder of this paper will be divided into four sections: The first will discuss errors made by students deriving from misinterpretation of problems; in the second section I will discuss how problem solving performance depends upon the verbosity of the problem statement; the third section will deal with logic problems involving double negatives as well as how adding certain biases to the problems affects student performance; the last section will contain the concluding remarks.

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ERROR TYPES ATTRIBUTED TO MISINTERPRETATION

The assumption that a non-native speaker of the English language will be as facile as a native speaker in understanding a mathematical word problem may not be valid. Subtleties in sentence construction, jargon, etc. increases the likelihood of misinterpretation for bilingual students. It has been my experience that Hispanic bilinguals often misinterpret problems despite the fact that they are familiar with all the vocabulary used in the problem. In these cases, a student will necessarily translate a problem into mathematical terminology incorrectly, yet totally consistent with the misinterpretation. A few examples will illustrate the situation.

The following problem revealed two interesting misinterpretations:

> In an engineering conference, nine meeting rooms each had 28 participants, and there were 7 participants standing in the halls drinking coffee. How many participants were at the conference?

This problem resulted in an error rate of 47% for the Hispanic group and 27% for the Anglo group. The incorrect answer I would like to discuss consists of students writing 9(28)-7=245. Two misinterpretations appear to be present here. First, these students interpreted "participants" to mean "those present at the meeting rooms listening to presentations", as opposed to the intended meaning of "all those who registered at the conference no matter where they happen to be at the time of the presentations." The second misinterpretation consists of assuming that the coffee drinkers came out of the nine meeting rooms. Thus, these two misinterpretations working in unison will logically lead to the answer 245. This error accounted for 50% of all Hispanic errors and 26% of all Anglo errors.

The error rate in the following problem

> A carpenter bought an equal number of nails and screws for $5.70. If each nail costs 2¢ and each screw costs 3¢, how many nails and how many screws did he buy?

for Hispanics and Anglos was similar, namely, 57% and 42% respectively. One type of error, however, was made almost exclusively by the Hispanic group and accounted for 35% of all errors. It consisted of interpreting the first sentence of the problem to mean that an equal amount of money was spent on nails and screws out of the total $5.70 spent. This interpretation resulted in the carpenter buying 95 screws and 142½ nails.

Other language related errors were found in the following problem:

> Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university". Use S for the number of students and P for the number of professors.

Disturbingly many mainstream engineering students commit the famous "variable reversal" error, $6S=P$, when working out this problem; the nature of this error has been discussed extensively in the literature (Clement, 1982; Clement et al., 1981; Rosnick, 1981). Our investigations with Hispanic students on this problem revealed erroneous answers which were virtually nonexistent among Anglos. For example, some Hispanics wrote $6S=6P$. Others wrote $6S+P=T$. Clinical interviews revealed these students' rationale for writing these answers. Those who wrote $6S=6P$ explained that the phrase "as many students as professors" meant an equal number of each; that is, $S=P$. The "six times" in front of the statement was interpreted to mean that each side of $S=P$ should be multiplied by 6. Those who wrote $6S+P=T$ explained that this equation related the number of students, professors, and the total student-professor population, T, in the appropriate proportion. Further details of this study can be found in Mestre et al., (1982).

PROBLEM SOLVING PERFORMANCE AS A FUNCTION OF VERBOSITY

In an attempt to examine how performance in solving algebra word problems is influenced by the amount of semantic processing required by the problem, four tests were designed. The four tests, Terse Word Problems and Verbose Word Problems, each in both English and Spanish, were constructed so that each of the two Terse-Verbose pairs are completely equivalent in mathematical content. However, in the Terse tests the questions are brief and simply stated, while in the Verbose tests the questions were embellished with irrelevant technical jargon. The Spanish versions were given only to the Hispanic group, while both groups took the English versions. The Terse and Verbose tests were not administered contiguously, since students would be likely to detect the similarity in content between the pair. As can be seen from the following Terse-Verbose equivalent problem pair, the problems required only elementary algebra since the intent was to study the degree to which semantic processing could affect performance:

> A number divided by a second number gives 2. The first number minus the second number also results in 2. What are the numbers?
>
> The Gartenhaus-Schmidt stress quotient is defined to be the loss of sleep factor divided by the coefficient of fear. The quantity obtained

> by subtracting the coefficient of fear from the loss of sleep factor is given the name Breakpoint Indicator. What are an individual's loss of sleep factor and coefficient of fear given that his Gartenhaus-Schmidt stress qoutient is 2 and that his Breakpoint Indicator is also 2?

The results of these tests are shown on the following table where the maximum possible score was 10 (one point assigned for totally correct responses, and 0 for any erroneous response):

	HISPANIC N = 60				NON-MINORITY N = 73	
	Spanish		English		English	
	Mean	Sta.Dev.	Mean	Sta.Dev.	Mean	Sta.Dev.
TERSE	6.87	2.01	7.82	1.99	9.26	.94
VERBOSE	4.40	2.39	6.45	2.51	8.56	1.30
D = TERSE-VERBOSE	2.47*	1.72	1.37*	1.44	.70*	1.28

$^*p < .01$

As the table shows, the performance of both groups was poorer on the Verbose test. All three difference-means, D, were significant beyond the .01 level, with the Anglo group being less affected by increasing the verbosity. The difference in the difference-means between the Anglo group and the Hispanic group in English is also statistically significant beyond the .01 level. This is indicative of the adverse effect that language deficiencies have upon mathematical performance in tasks that require substantial semantic processing. A possible explanation for the disparity in performance across language for the Hispanic group is that these students have received most of their mathematics and science instruction during high school and college in English and as a result, may not be very facile at solving word problems in Spanish.

EFFECT OF DOUBLE NEGATIVES AND BIASING IN LOGIC PROBLEMS

The final study I would like to discuss is one we are currently in the process of analyzing. It involves a topic which, although not directly related to mathematics, is nevertheless relevant to problem solving. The topic is logic problems containing double negatives. I originally became interested in double negatives because the logical structure of the Spanish language allows certain doubly negated statements to remain negative, instead of reverting to a positive statement, as would be the case in the English language. For example, the grammatically correct translation of the statement "I do not want any money" into Spanish is "yo no quiero ningún dinero", which if translated literally into English becomes "I do not want no money".

I thus wanted to find out if this lack of parallelism between the two languages would cause confusion to Hispanic technical college students.

To investigate this question, we developed a 20 question test of 2-choice answers with problems similar to the one following:

Nobody was unbeatable by Jennifer at the track meet means:
a) Jennifer beat everybody at the track meet.
b) Jennifer did not beat everybody at the track meet.

This test was administered to three groups of students: 26 Hispanic science majors, 78 Anglo science majors, and 47 Anglo non-science majors who were taking a "remedial" math course. The means based on total number correct for these three groups respectively were 16.8, 18.5, and 16.9. Defining mastery as those scoring 17 or above, we found that 38% of the Hispanics and 36% of the non-science-majoring Anglos had not mastered these types of double negations, compared to 6% of the Anglo science majors.

On slightly more difficult logical double negative tests, we found (not surprisingly) that Anglo science majors outperformed Hispanic science majors, but that the Anglo group's overall performance was dismal. Further, on problems which were purposely "biased" such that the correct answer was in direct conflict with the "real world", we found that Anglos and Hispanics alike performed significantly lower than in totally equivalent unbiased problems. Perhaps an example of what I mean by "bias" would clarify the discussion. Consider the following two problems:

1. If it is not true that not all college graduates earn less than $7,000 a year, then it must be true that:
   a) Some college graduates earn more than, or equal to, $7,000 a yr.
   b) All college graduates earn less than $7,000 a year.
   c) All college graduates do not earn less than $7,000 a year.
   d) Some college graduates do not earn less than $7,000 a year.
   e) All college graduates earn more than, or equal to, $7,000 a yr.
2. If it is not true that not all fig trees grow more than 9 inches per year, then it must be true that:
   a) All fig trees do not grow more than 9 inches per year.
   b) All fig trees grow less than, or equal to, 9 inches per year.
   c) All fig trees grow more than 9 inches per year.
   d) Some fig trees do not grow more than 9 inches per year.
   e) Some fig trees grow less than, or equal to, 9 inches per year.

The first problem is biased, the reason being that the problem's answer, b, is contrary to what is actually true in the real world. On the other hand, the second problem contains no such bias since people's knowledge about the yearly growth of fig trees is likely to be limited.

Two tests were developed with 15 questions such as those above, one in Spanish, the other in English. Out of these 15 questions, 10 were structurally equivalent by pairs--5 of them biased, and 5 unbiased. We also wanted to study the effect of additional negations upon performance. To do this, 14 of the questions were constructed structurally equivalent by pairs, with one group of 7 questions containing one more negation than the other group of 7. For example, the problem equivalent to #1 on the previous page with an added negation read: "If it is not true that not all sharks are not more than 8 inches long, then it must be true that:". The following table summarizes the results of this logic test:

MEANS AND STANDARD DEVIATIONS ON LOGIC TEST

Group and Language of Test		Total Score	Effect of Bias		Effect of Extra Negative	
			Unbiased	Biased	Neg.	Neg.+1
Hispanics (N=49 technical majors)	Spanish	5.18 (2.39)	1.67 (1.18)	1.24 (1.15)	3.12 (1.71)	1.92 (1.12)
	English	4.80 (2.15)	1.82 (1.21)	1.35 (0.92)	2.45 (1.34)	2.02 (1.22)
Anglos (N=53 technical majors)	English	7.19 (2.55)	2.68 (1.16)	2.28 (1.32)	3.89 (1.37)	2.77 (1.45)
Max. Possible Score		15	5	5	7	7

Several things are clear from this table. First, the overall performance of the Anglo group was poor, although better than the Hispanic group's performance in both English and Spanish; the difference in overall performance between the Anglo group and the Hispanic group in English is statistically significant well beyond the .001 level. Second, it is clear that in going from unbiased to biased problems, both groups' performance level dropped--in fact, all three difference-means are statistically significant beyond the .05 level. Similarly, the performance level of both groups dropped when one additional negative was added to a set of problems; again, all three difference-means here are statistically significant beyond the .05 level.

DISCUSSION AND CONCLUDING REMARKS

The results we have presented show that language skills and problem solving skills are not decoupled; in fact, indications are that problem solving skills are largely mediated by language skills, particularly for students such as the Hispanics of this study who are not as proficient in English as native speakers. Further support for this claim can be found in a recently completed study where we found strong correlations between various mathematics tests and language proficiency measures for a group of Hispanic technical students (Mestre, 1981).

It is my feeling that integrating language and problem solving would be of benefit not only to students with language deficiencies, but to all students. This integration would not require large modifications in our current teaching styles. For example, more emphasis and drill work in "setting-up" problems would make students more adept at quickly gleaning the important information from a problem and translating this information into mathematical terminology, while at the same time allowing the instructor to discuss misinterpretations as they arise. Instructors could also emphasize the importance of language precision in communicating technical information, and illustrate how sloppiness can result in ill-defined problems. Since students seem to have a proclivity to draw from their preconceptions or intuitions in solving problems, instructors could warn students against being misguided by their "gut reactions", and be critical of the procedures they use in getting answers. These and other techniques such as checking the answer with an independent method, using diagrammatic techniques in facilitating solutions, etc., should add another dimension to students' problem solving repertoire.

I would like to conclude by bringing two points to the attention of those of us who may be teaching students with below-average language skills. First, we should be aware that the message received by these students may often not be an accurate representation of the message sent. Since it is not unlikely that we will use double negations, awkward constructions, etc., in our lectures and exams, there is a nontrivial probability that these students will interpret our messages inaccurately. Second, in exam situations, a student with underdeveloped language skills will necessarily take longer to read and understand the questions, and thus may experience a shortage of time during which he/she can display his/her expertise.

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THREE PERSPECTIVES ON THE PROCESS UNIFORMITY OF BEGINNING ALGEBRA STUDENTS[1]

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In a recent study on the learning of algebra, a majority of students persisted in solving various algebra problems in a rulebound, algorithmic fashion despite far simpler solution processes suggested by the problem structure. In this paper one example of this kind of process uniformity, the "single letter fixation," is analyzed within the context of three theoretical frameworks—information processing theory, meaningful learning theory, and cognitive development theory. The results obtained from the clinical interviews of eight high school freshman algebra students are examined and possible implications for the teaching and learning of algebra are discussed.

One striking feature of the problem-solving processes of first year high school algebra students seems to be their uniformity of response. The general lack of flexibility in students' solution processes becomes particularly noticeable when virtually everyone who is interviewed follows a relatively complex sequence of routine algorithms to solve a problem with a trivially simple solution. For example, in our recent NIE-funded research project on the learning of algebra, one question we asked in our clinical interviews was: "Find the value of the expression $\frac{2x+1}{2}$, given that $5(2x+1) = 10$." The predominant procedure that the students followed was to solve the equation for x, substitute the value of x into the expression $\frac{2x+1}{2}$, and thus arrive at the answer 1. Clearly, this method "works," but it seems rather circuitous in view of the fact that the question merely asks for one-half the value of the expression $2x + 1$, which by inspection of the equation must be 2, thus making the answer 1.

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This tendency for students to solve for the value of a single letter when working on problems that can be handled more efficiently by finding the value of a binomial expression directly is part of a phenomenon Assad (1978) calls the "single letter fixation." In this paper we will analyze the single letter fixation phenomenon from three theoretical perspectives: information processing theory, meaningful learning theory, and cognitive development theory. In addition, we hope to demonstrate the usefulness of considering multiple theoretical perspectives in generating ideas and suggestions for the teaching of high school algebra.

STUDENT RESPONSES

When the eight ninth-grade algebra students in our study were asked the aforementioned question we found that the only successful solution paths that they followed were the two described above. Schematically these solutions can be represented as follows:

Method A:

$$\begin{aligned} 5(2x + 1) &= 10 \\ 10x + 5 &= 10 \\ 10x &= 5 \\ x &= 1/2 \end{aligned} \qquad \therefore \quad \frac{2x+1}{2} = \frac{2(\frac{1}{2})+1}{2} = \frac{1+1}{2} = 1$$

Method B:

$$\begin{aligned} 5(2x + 1) &= 10 \\ 2x + 1 &= 2 \end{aligned} \qquad \therefore \quad \frac{2x+1}{2} = \frac{2}{2} = 1$$

Six students used method A, one used method B, and one was unable to devise a systematic solution procedure, but instead guessed an answer of 4. It should be noted that no student who chose method A deviated in the slightest from the steps shown above by, for example, dividing $5(2x + 1) = 10$ by 5 instead of "clearing the parentheses" first. This near unanimity of response seems to suggest a rulebound, algorithmic approach to the solution of the equation. It is the students' behavior in answering this question that we wish to explain from three theoretical perspectives.

INFORMATION PROCESSING PERSPECTIVE

Within the context of information processing theory we will consider three possible explanations for the single letter fixation. First, it could indicate that the learner's "executive control" is lax in searching among available solution procedure options when a routine algorithm is available. That is, the student, when presented with a task that involves solving an equation, may just follow a standard, taught procedure rather than look for a shortcut.

A second information-processing explanation for the single letter fixation might be that the learner does not possess an alternative solution procedure to call upon. One alternative, and shorter, solution procedure for the given problem would involve the implicit or explicit substitution of a single letter for the binomial $2x + 1$ in the given equation. This binomial substitution approach, coupled with a concept of transitivity, would lead in only one additional step to the conclusion that the value of $2x + 1$ is 2, and hence one-half of it is 1. If no such alternative procedures are available to the student, then the persistent use of method A might be more accurately attributed to the paucity of available procedures than to laxity in the executive control.

Still a third information-processing explanation is possible, that involving the notion of a "visually-moderated sequence" (Davis, Jockush, & McKnight, 1978). According to this idea it is hypothesized that a visual form precipitates a subconscious action whose output in turn triggers another action, with this process continuing iteratively until some terminal stage is reached. The theory suggests, for example, that the equation $5(2x + 1) = 10$ sets the standard "equation solving" algorithm in motion and, once begun, each step triggers the next, so that the student is automatically led through the steps of the method A solution procedure, in a process vaguely reminiscent of Thorndike's (1926) bond theory. To whatever extent this interpretation of a student's behavior has validity, the overemphasis on drill in a standard algorithm may be robbing the student's "executive control" system of its principal role as process overseer.

If flexibility of approach is a desirable characteristic of problem-solving behavior, and we believe that it is, these three information-processing interpretations of the single letter fixation suggest a number of implications for the teaching of algebra. First, teachers should, by word and example, model the "executive control" operation of considering options throughout the course of problem solving. Once students have achieved a certain minimum facility in working problems of a certain type, it behooves us to demonstrate alternative solution strategies if we, in fact, wish the students to exhibit forethought and creativity in their own solution approaches.

Second, students should be instructed in the existence and proper usage of general algebraic procedures that cut across problem types, like the technique of binomial substitution. It seems unwise to assume that these general procedures come naturally to most students. Yet, the very fact that these procedures are not problem-specific means that they may often be overlooked in the curriculum.

Third, while there are certain aspects of mathematics, such as multiplication facts, to which students should ultimately react automatically, it is questionable whether there is much material in the learning of algebra that should be reduced to this level. Algebra is, in a very real sense, generalized arithmetic and should, by its very nature, teach students an appreciation for structure and pattern. Student "exercises" should be varied in such a way as to stimulate consideration of options. Too often, textbook exercises are so much alike that they serve only as a drill for the reinforcement of algorithmic procedures. When even the variations in the exercises become predictable, students are denied the opportunity to look beyond the superficial aspects of problems to gain a deeper appreciation for the essence of algebra.

MEANINGFUL LEARNING PERSPECTIVE

In the context of meaningful learning theory the single letter fixation suggests that the learner may not realize that an expression like $2x + 1$ can be considered a variable (symbol) just like x or y. Thus, the single letter fixation might be an indication that the learner's concept of variable is incomplete. An analysis of the student's exposure to literal symbols helps to explain why a limited conceptualization might develop. Because students generalize from their personal experiences, and textbooks require them to solve equations only for single letters, it is not surprising that students do not think of solving an equation for a binomial expression.

To broaden students' exposure to variables in different forms, teachers could assign some equations in which the student is instructed to solve for expressions like $2x$, $-y$, and $3n + 2$. These exercises should contain equations in which the specified expression can be solved for directly, if treated as a single entity. These exercises would provide a natural motivation for an open discussion of the single letter fixation with the class. The discussion would serve to create a student awareness of this tendency, with the desired outcome being increased sensitivity and flexibility in handling variables.

The single letter fixation is just one facet of the many difficulties students experience with the concept of variable, but it can serve as a warning flag to teachers that additional work on this concept is needed. More varied experiences with variables during the early stages of learning algebra could help to correct or even stave off other common misconceptions about literal symbols, such as "$-x$ represents a negative number" or "x, y, and $2x + 1$ must represent different values since they use different letters." Certainly a more meaningful understanding of the concept of variable must be a primary objective of any elementary algebra instruction.

COGNITIVE DEVELOPMENT PERSPECTIVE

From the perspective of cognitive development theory we can suggest two plausible explanations for the single letter fixation. First, when this fixation occurs in the context of equations, it may be related to the student's understanding of the equal sign. In the early grades students interpret the equal sign operationally as a command to perform the operation indicated on the left of the symbol and record the answer on the right. It is only much later that students begin to understand the equal sign structurally as a symbol for an equivalence relation (Kieran, 1981). The single letter fixation may be evidence that a student is still in a period of developmental transition with respect to understanding the equal sign. The "do something" command of the equality symbol may still be so strong in the student's thinking that, upon seeing an equation like $5(2x + 1) = 10$, the student automatically reacts operationally and begins solving the equation without ever noticing the structural similarity between the $2x + 1$ expression in the equation and the $\frac{2x + 1}{2}$ expression to be evaluated.

A second developmental explanation for the single letter fixation draws a parallel between the problem-solving behavior that we observed in algebra students and the problem-solving behavior that others have observed in young children. We found that many algebra students either did not recognize shortcuts or did not feel comfortable using them, even when these shortcuts were built into problems in the form of hints. The inability and/or reluctance of algebra students to use shortcuts may be analogous to the counting-all behavior that persists in some young children even when they have been shown the counting-on shortcut (Resnick & Ford, 1981). That is, until the child's thinking is developmentally prepared to accommodate to a shortcut, the shortcut is essentially not assimilated. The single letter fixation might be considered as evidence that many algebra students are not ready to assimilate the structure of algebraic expressions well enough to use shortcuts based on that structure.

These two developmental explanations for the single letter fixation each have implications for teaching. One implication is that teachers should not treat the relational properties of equality superficially under the assumption that these ideas are trivial. Students need to be weaned from their operational interpretation of the equal sign through exposure to problems in which use of the reflexive, symmetric, and transitive properties of equality is essential.

Another implication for teaching relates to the use of shortcuts. There may well be a developmental factor involved in students' willingness to look for and use shortcuts. However, it may also be that many teachers discourage creativity by insisting too strongly that students show every step of their work. Perhaps the valid concern that students learn standard procedures can be counterbalanced with increased appreciation for student insight and positive encouragement of creative efforts.

CONCLUSION

Examination of the single letter fixation from three theoretical perspectives has helped us gain a clearer understanding of the various factors that may contribute to this phenomenon. Moreover, in considering all three perspectives, we have been able to generate many more ideas for instruction than we would from considering only one theoretical perspective. Since one of the earmarks of a good problem solver is flexibility, then we as mathematics educators, in our role as problem solvers in mathematics instruction, could probably benefit from more use of multiple theoretical perspectives in our own hypothesis formulation.

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A TEACHER'S ANALYSIS OF STUDENTS' PROBLEM-SOLVING PROCESSES IN ALGEBRA*

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The following paper presents the observations and suggestions of a teacher with regards to a clinical study of the processes used by four of his students in solving algebra problems. Each student was videotaped individually in seven one-hour interviews conducted during the school year. The students' algebra teacher aided in the analysis of the interviews by providing his personal insight from two years of instructing the students. Three goals for this analysis were: to learn which problem-solving procedures are similar to those used or anticipated by the classroom teacher, which errors are similar to those made in class and to find which procedures and errors can be explained by classroom or textbook presentations or specific student background information. In addition to these goals the teacher also utilized his review of the interviews to provide suggestions for teachers of introductory algebra and a critique of the conduct of the study. The analysis procedure had four steps: the teacher solved the same interview problems as the students, the teacher predicted students' solutions, the teacher analyzed students' solutions, and finally, the teacher discussed observations and suggestions with the interviewer. At the end of the eighty hours of analysis the teacher collected his thoughts into two papers, Hints for Teachers of Introductory Algebra and Suggestions for the Researcher.

Most research on the learning of algebra has focused on the products of the mathematics curriculum. The ability of the researcher and teacher to predict product measures has shaped how we teach mathematics. Recently there has been growing interest in investigating the processes involved in the learning of algebra. These studies have emerged largely from three different sources: information-processing theory (Carry, Lewis, & Bernard, 1980; Davis, 1975a, 1975b; Davis & McKnight, 1976, 1979; Matz, 1979), Piagetian theory of cognitive development (Collis, 1975; Firth, 1975; Johannot, 1974; Küchemann, 1978; Wagner, 1977), and Vygotskiian theory of cognitive development (Krutetskii, 1976; Rachlin, 1982). One common feature across these studies is the development of models for predicting the processes used by algebra students. Although most of these models have been created through

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clinical interviews with learners, the interviews and models have rarely been analyzed by the learners' classroom teachers.

In an effort to balance their observations of processes used by students in solving algebraic problems with the teacher's perspective from the daily classroom activities, Rachlin and Wagner have included a teacher analysis as an integral phase of their own analysis in two studies of the problem-solving processes used by algebra students. The first study (Jensen, Rachlin, & Wagner, 1982) was conducted in Athens, Georgia with 10 ninth-grade algebra students. Each student was interviewed for 45-minutes every three weeks over the course of the academic year, using the Soviet ascertaining methodology (for a description of the methodology see Wagner & Rachlin, 1981). The second study (Rachlin & Wagner, 1982) was conducted in Calgary, Alberta, Canada with 4 ninth-grade algebra students. It replicated the first study with each student videotaped approximately once every two weeks from February through May. In the summer of 1982, the students' algebra teachers participated in the initial analysis of the students' protocols. Although the procedures developed were common to both sites, Calgary and Athens, the present paper is limited to the teacher's analysis of the study conducted in Calgary.

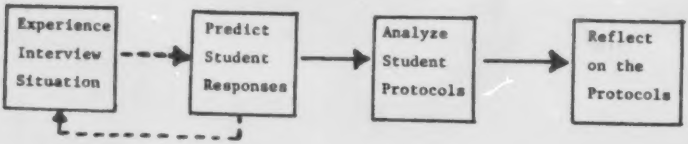
GOALS FOR MEETING WITH CLASSROOM TEACHERS

The inclusion of the classroom teacher in the analysis process was guided by five objectives agreed upon by Rachlin and Wagner:

1. To learn which problem-solving procedures are similar to those used or anticipated by the classroom teacher and which are different;
2. To learn which student errors are similar to those made in class and which are different;
3. To find explanations for procedures and errors based on individual student background information, classroom instruction, or textbook treatment of a topic;
4. To generate suggestions for teaching introductory algebra;
5. To critique the conduct of the project.

ACTIVITIES FOR MEETING WITH CLASSROOM TEACHERS

In order to meet the above goals, a series of activities were developed by Rachlin and Wagner. Four activities were repeated in nine sessions (including a pre-test, post-test and seven series of interview tasks) as indicated in the following model:


Experience Interview Situation
Predict Student Responses
Analyze Student Protocols
Reflect on the Protocols

First, the teacher was interviewed by the researcher. Since the video-tape facility used with the students was not available, the interview sessions were audio-taped for later analysis. The interview was given using the same General Suggestions for the Interviewer, Interviewing Strategies Specific to the Study, Suggestions for Handling Errors, and Guides to the Interviewer (specific to each task) used to direct the student interviews (see Rachlin and Wagner, 1982 for detailed descriptions of each instrument).

After completing each task in an interview, the teacher predicted each of the four student's solution processes for the task. This included how the student would proceed, how the student's solution would differ from his, and how difficult the task would be for the student. This discussion included variables which might effect the student's solution process; e.g. the textbook, class presentation, and specific student background information. When the teacher completed his predictions of the students' solution processes for the given task, he was given the next task in the interview session.

Once this process was completed for an entire interview session, the teacher reviewed the video-taped interviews that had been conducted with his students. A form entitled the Teacher's Review of Student Protocols was used for each interview. On the top two-thirds of the form the teacher placed comments and/or explanations relative to student errors and procedures. On the bottom third of the form the teacher noted instances where the text, classroom presentations, or specific information on the students might have effected the student's solution processes. In this area the teacher also indicated any surprises which he experienced in reviewing the protocols and any suggestions he might have for other teachers based on the interview. The back of the form was used for the overflow of written notes.

After reviewing all four students' protocols for the interview session, the teacher was again interviewed by the researcher. This interview served as a debriefing session. It provided an opportunity for the teacher to talk to the researcher about his observations. This ranged from specific comments on an individual student's solution to a particular task to more global concerns on teaching in general. This interview was largely directed by the teacher. After completing this reflection phase, the process was begun anew on the next interview series.

Once all of the interview sessions (including the pre-test and post-test) were completed in this fashion the teacher was asked to write a short list of suggestions for teaching beginning algebra and a short critique of the project with some suggestions for improving the research methodology.

COMMENTS RELEVANT TO THE PROCESSES UNDER INVESTIGATION

Although the teacher was never explicitly told that the specific problem-solving processes used to guide the construction of the interview tasks were reversibility and the ability to generalize, it did not take long for him to guess the directions of interviewer questioning. Upon being given a task such as "Find two fractions whose product is greater than 1.", he initially created a general solution and then gave a specific one. He then tried to guess what follow up questions the interviewer might ask. For example, in solving the above task, he anticipated that the interviewer would also ask for two fractions whose product was less than one and so he gave the general solution to that problem as well as the given problem. In his review of his students' work, the teacher indicated that he was pleased with two aspects of their ability to generalize: the students creation of a simple problem as a heuristic for solving a given problem and their desire to find patterns in their solutions for problems. He commented that he had placed a great deal of stress on these processes throughout the school year.

In contrast to his evaluation of his students' ability to generalize, the teacher was not as positive with regards to the students' reversibility. The teacher was very flexible in his solution of tasks such as "What number added to the sum of 17 and 6 equals 6?" and solved it in several different ways but this flexibility was not reflected in his students' solutions. Although the teacher predicted that the students would all quickly apply a compensation approach to this task (noting the sixes in both members), he was surprised to find that only one student used this approach. Instead, the others used negation to solve the task (first adding 17 and 6 and then subtracting 23 from 6). Despite encouragement to find other ways of solving the problem, it was not until the interviewer directed the students to check their answers in the original task that two of the students noticed the relationship.

In his Hints for Teachers of Introductory Algebra, the teacher explained, "I do not spend enough time doing questions of the sort A * ___ = B or ___ * ___ = A, and having the students provide answers once I've specified the conditions on the answers. I think my students (maybe yours, too) get conditioned into looking at the 'answer' as being the thing on the right." The teacher went on to extend this discussion to the solving of equations. "In solving equations my students viewed the right hand side as being the place for the answer. I think this emphasis on that particular form must have come from me, and yet what I had intended was that they simplify equations to the form $x = a$ (or $a = x$) so that they would know what to replace x with to make a true statement. Sometimes (maybe very often) what we teach is not what they learn."

One example where a specific procedure taught in class may have effected the processes of reversibility used by the students, was given by tasks such as "What binomial multiplied by $x + 5$ equals $2x^2 + 15x + 25$?" Although the teacher used a form of compensation to solve the task, writing $(x + 5)(\quad) = 2x^2 + 15x + 25$ and then solving for the unknown binomial, his students tended to ignore the given binomial

$x + 5$ and to solve the problem by factoring $2x^2 + 15x + 25$ using a procedure which they learned in class. They wrote down $\_\_\_ + \_\_\_ = 15$ and $\_\_\_ \cdot \_\_\_ = 2 \quad 25 = 50$ and searched for values that would satisfy the two equations. After noting that 10 and 5 would work, they wrote the $2x^2 + 15x + 25$ as $2x^2 + 10x + 5x + 25 = 2x(x + 5) + 5(x + 5) = (2x + 5)(x + 5)$ and then determined that the missing binomial was $2x + 5$. None of the students solved the task by reversing their learned procedure; i.e., noting that $\_\_\_ \cdot (x + 5) = \_\_\_ (x + 5) + \_\_\_ (x + 5) = 2x^2 + 15x + 25$ and then compensating to find the missing terms. On other tasks involving factoring polynomials with four terms, three of the four students indicated that they had never solved anything like that before (despite the fact that it was a step in their factoring procedure).

In his Hints for Teachers of Introductory Algebra the teacher noted, "I was taught to factor trinomials by trial and error; e.g., $6x^2 + 19x + 15 = (\quad + \quad)(\quad + \quad)$, and then I learned to teach factoring of trinomials by the sum and product method; e.g., for $6x^2 + 19x + 15$ we need $a + b = 19$, $a \cdot b = 90$. I had assumed that this was superior. Now I think both have a place in my classroom. There are times when the trial and error method is faster (probably not in the case just given), and that can be useful. Form doesn't always have to be important."

OTHER COMMENTS

Another area in which the teacher believed that the difficulties experienced by the students may be a reflection of his classroom instruction was on tasks involving fraction. Before ever beginning a task involving fractions, several students would verbally or physically indicate a negative reaction to the task. In his Hints for Teachers of Introductory Algebra the teacher noted, "I've got to find more and better ways of dealing with teaching fractions so that my students see a little joy in them, or at least less distaste. I think that their dislike for fractions gets in the way by creating mental blocks when fractions are involved in other problems and questions."

Not all observed procedures could be explained by classroom instruction. For example, the teacher was surprised that three of the four students solved tasks such as "What trinomial subtracted from $5x^2y - xy^2 + 7$ equals $-x^2y + 8$?" by writing the parts vertically. With regards to the vertical form the teacher commented, "They've seen it occassionally in the textbooks, but I've never assigned the assignments that have vertical form." In his Hints for Teachers of Introductory Algebra the teacher added, "Several years ago I adopted the horizontal form of adding and subtracting polynomials as being the best, and eschewed the vertical form as being a creator of difficulty. I've changed my mind after watching my students perform on videotape. Now $\begin{array}{r} ax + b \\ + \underline{cx + d} \end{array}$ appears to be more 'natural' for them; they repeatedly used this form when 'I wasn't looking' and with good success, too."

The triad of the curriculum, the teacher, and the learner is at the core of any educational research. The curriculum can be defined by the products and processes of each specific content area. The teacher (or author or programmer) presents the curriculum to the learner. It is the teacher who interprets the curriculum for the student. Given a particular curriculum, the ability of the teacher to predict student responses and to modify instruction accordingly is essential for individualizing instruction. How teachers interpret the curriculum is central to any comparisons of methodologies. The educational significance of a research study may in the long run be dependent on the ability of an investigator to clearly describe the interaction of the components of the triad. The present study is an attempt to provide an important connecting link in the study of the problem-solving processes of algebra students.

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A MODEL OF CONCEPT UNDERSTANDING FOR SECONDARY SCHOOL MATHEMATICS

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Recent efforts have been made towards the formulation of models of understanding in mathematics, and particularly focusing on the understanding of mathematical concepts. These models are potentially important instruments for the researcher and some attention may be given to their refinement and improvement. This paper attempts to present a model especially conceived to conceptualize students' understanding of functions.

There is no single model of understanding accepted by all mathematics educators (Byers, 1980). Classical models of psychology that deal with concept learning focus on the production of examples, non-examples, and definitions. Two main traditions can be recognized. One, following Bruner, gives attention to the abstraction of the concept through discovery from examples and non-examples (concept attainment). The other, following Ausubel, gives priority to learning from the verbal statement of a definition and the discussion of the relevant attributes of the concept (concept formation)(Sowder, 1980).

During the first half of the twentieth century the use of an all-or-nothing dichotomy was common in mathematics education: the student either understands or does not understand. The focus was on the apprehension of a whole theory rather than on the mastery of a single specific concept. More recently, starting with the work of Bloom and his colleagues (1956) attempts have been made to set up more complex models. Skemp's (1976) characterization of relational and instrumental understanding was very influential in the further developments that have been undertaken in the last few years.

Sowder (1980) describes a model with five sequential levels (concrete, identity, production, classificatory, and formal) that, with minor adaptations, is meant to be applicable both to concept and principle learning.

Herscovics and Bergeron (1981; Bergeron and Herscovics, 1982) presented an elaborated model of understanding with four levels arranged in hierarchical sequence (intuitive understanding, initial mathematization, abstraction, and formalization). They applied this model to the concept of function describing levels involved in the building of this notion taken as a conceptual schema. This model is meant to account both for the processes and the results of construction of the concept.

DIMENSIONS OF UNDERSTANDING

The model here presented refers to the understanding of mathematical concepts. With Herscovics, Bergeron and Sowder, I find it useful to describe levels and dimensions of understanding in operational terms. The following are proposed as dimensions (or contexts) of understanding a concept:

First dimension (apprehension): The student discriminates among examples and non-examples. This ranges from simple situations and the most typical cases to more complicated and subtle ones. The student can also be asked to give new examples and non-examples.

Second dimension (characterization): The student can reproduce a suitable definition in his or her own words or give an acceptable characterization of the concept.

Third dimension (use): The student can use the concept to solve problems, ranging from simple exercises to open-ended problems.

Fourth dimension (system): The student is able to insert the concept in a system with other concepts and relations among concepts (axioms, theorems, procedures).

It should be noted that these dimensions are not arranged in any linear or hierarchical order. Use and system are very essential and unfortunately often forgotten dimensions. They remind us that concepts, being fundamental mathematical elements, do not have worth by themselves but as a part of a whole. They depend on a whole, which has strong implications for the understanding that the students may or may not attain.

Several factors may contribute to the learning and the understanding of a concept, such as the statement of a definition or the production of a set of examples and non-examples. But one factor that by no means must be disregarded is its use."I do and I understandand" is also true in concept understanding.

No mathematical concept is created without a specific aim. It must satisfy some purpose: to provide a model for some practical situation or problem, to help one to formulate a mathematical result concisely, or to allow one to state a new relationship. The greater the usefulness of a concept the greater its relevance. The grasp of the relevance of a concept is an important factor in its overall understanding by the student.

LEVELS OF UNDERSTANDING

The development of a concept (and especially of a basic fundamental concept) is a long term process. As Van Engen (1953) put it: "Conceptual development is a growth process. It takes time. . . . The teacher should not expect the pupil to develop a concept in a few days" (p.94). Sowder (1980) also recognized that a student may form a richer concept by working with it for a long time. He says that "much of this richness derives from experience with the concept . . . in the context of other concepts and principles" (p.277).

Every dimension embraces a wide range of gradations and interconnects with the remaining dimensions. Four levels will be distinguised in each of the mentioned dimensions of understanding, and here some similarities will be noted with the model of Bergeron and Herscovics.

(1) Students begin with perceptions, intuitions, and some pre-requisite knowledge. This can be called a pre-concept level. (2) As a result of classroom work, they may pass to a primitive level of "initial mathematization" where they meet typical reasonings, relationships, procedures, and computations that they assemble into a single notion. The presence of a suitable label may play an important role helping the student to organize in his mind these primitive elements. (3) Then, an abstraction process can occur, and the result is a clearer identification of the concept. However, even at this third level many difficulties may still exist in working with the concept in some settings. (4) The final overcoming of these difficulties and the ability to handle the concept as an object in a higer level theory characterize the mature level.

The intuitions of the students develop from level to level. And, despite natural differences, there are necessarily formal elements at each level, since content does not exist without form , nor form without content. These are the reasons why I think that it is potentially misleading to call "intuitive" the most primitive level,and "formalization" the most advanced one, as Bergeron and Herscovics (1982) and other authors tend to do. Another term that should be avoided as the label of one level is "symbolism." It is almost impossible to work in mathematics without the use, in one way or another of symbols. So, symbols have to exist from the beginning, and may be developed from level to level.

Despite the strong similarities of this model to that of Bergeron and Herscovics, some differences should be noted. Those differences do not concern only the labels of each level: the present model is clearly pluridimensional, is more classroom oriented, and views in a very different way the role of intuition, formalization, and symbolism.

PEDAGOGICAL AND RESEARCH QUESTIONS

The fourth level of understanding is the one of the professional mathematician or user of mathematics and it is not within the scope of the great majority (if not almost all) secondary school students. The goal of secondary school mathematics should be to bring students as close as possible to this fourth level.

Teachers need to know where their students are and what kind of activities are the most suited to develop with them. If the students do not have all the necessary prerequisites of the pre-concept level, this has to be built up before attempting the transition to the next level. If the teacher is forcing an abstraction process but the student has nothing to abstract from, certainly understanding will not take place. Understanding implies the restructuration of old pieces of knowledge. So it is unnecessarily made much more difficult if the teacher simply ignores the stage of the student.

If the classroom activities tend to focus only on one or two dimensions of the concept forgetting the remaining, it is very likely that the progress of the students is going to be slower than in a more balanced approach.

Do optimal treatments exist for the concepts of school mathematics? Which activities are most appropriate to develop with each kind of student?

It is possible that different approaches may lead to equally satisfying results. But it would be very useful to know the strong and weak points of each one as well as how to overcome the difficulties that the students find in their learning and understanding.

ACKNOWLEDGEMENTS

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DETERMINING REPRESENTATIONS OF CONCEPTUAL STRUCTURES IN TRIGONOMETRY

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Richard Skemp's theory of conceptual structures (schemas) was adapted by hypothesizing two types of schemas: (1) an underlying parent schema and (2) a problem solving schema which contains paths for problem solving. Three interpretations of the definitions of the trignometric functions were identified. For each interpretation, representative problems were chosen and methods of solution analyzed. From the methods of solutions the relevant concepts and specific actions employed were identified. The two schemas were then constructed for each interpretation. This preliminary model was tested and refined by interviewing four subjects. Interview analysis included construction of representations of the parent and problem solving schemas passessed by each subject in each interpretation. The refined model was then used to interview two additional subjects.

Interview analysis indicated strong face validity in that the model was found to contain the problem solving paths used by the subjects. The model was found to be a viable model of conceptual structures in which problem solving can be described and to have potential value for both instruction and research.

A conceptual structure has been defined as an "assemblage of identifiable [concepts] and the relationships among those [concepts] (Shavelson, 1974, p. 231). Such a construct is useful in describing not only the intellectual growth and development of learners but also their problem solving behavior. Since the ultimate concern of mathematics education is developing the problem solving skills of learners, it would be valuable to obtain representations of the conceptual structures within which problem solving can be described. Skemp (1971, 1976, 1979) has developed a general theory of goal seeking behavior which uses schemas (conceptual structures) as a vehicle to describe how goals are reached. It is his theory that serves as background for this work--goals are seen as solutions to problems and routes through the schemas are seen as problem solving paths.

For Skemp, as for many others, a schema is visualized as a network with vertices and paths connecting them. Such networks themselves have an ambiguity that allows them to be interpreted on many different levels. At one level, the vertices represent states and the paths indicate which states are obtainable from other states. At another level, the vertices represent concepts and the paths indicate how the concepts are related. At yet a third level, the vertices are seen as having interiority or a structure within themselves and the whole network represents the interrelationships among conceptual structures. Skemp refers to these levels as: a road map, a cognitive map, and a generalized schema (1979, pp. 167-168). To resolve this ambiguity for this study, two related schemas are hypothesized: (1) an underlying conceptual structure called a "parent schema"--it is thought of as static--and (2) a related structure called "problem solving schema" which represents the dynamic process of following problem solving paths through the parent schema.

The study had several purposes: (1) to adapt that portion of Skemp's theory that deals with conceptual structures to make it operational; (2) to develop, from that adaptation, a specific model of a structure of trigonometric concepts which contains paths for solving routine problems in trigonometry; (3) to use an interview process, developed from the specific model, to determine hypothetical representations of the conceptual structures possessed by subjects and to determine if the specific model contains the paths utilized by those subjects in problem solving; and (4) to generalize the processes of model formulation and interviewing in such a way that they can be used and tested other than in trigonometry.

PROCEDURES

The first step in the study was to decide on the content area for investigation. Trigonometry was chosen since it contains many complex interrelationships of concepts and is thus an ideal topic topic to be represented by conceptual structures. A preliminary model of trigonometric conceptual structures was developed from a study of methods of solving routine problems and an identification of the major concepts used in those problem solutions.

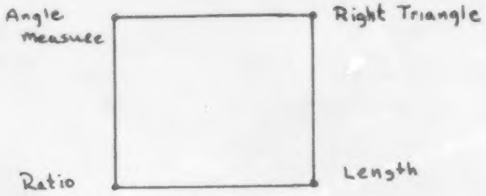
The preliminary model was refined after a series of four interviews with each of four subjects. The interviews were semisturctured and were based on the problem solving methods used to determine the preliminary model. Using the refined model, two additional subjects were interviewed. The interviews with one subject were analyzed by a second evaluator as a check on reliability. Three forms of validity (face validity, cross-validity, and comparison with word association data) were investigated.

PRELIMINARY MODEL DEVELOPMENT

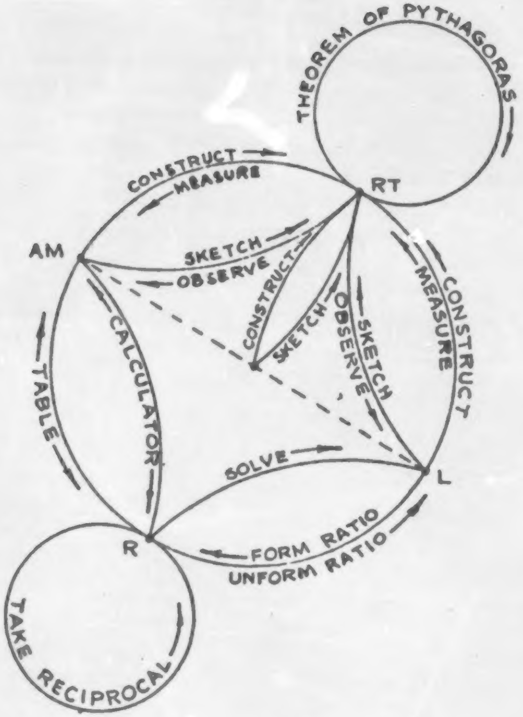
Since trigonometry is a very broad area, it was decided to limit the study to the use of construction, trigonometric tables, the calculator, and the analytic use of the definitions of the six trigonometric functions. Three different approaches (right triangle, coordinate system, and unit circle) were used with different schemas and interviews for each. Because of space limitations, only the right triangle approach will be described in detail in this report.

The goal of preliminary model development was to produce the two schemas for each interpretation and represent those as networks. Two very general principles guided the development: first, all actions should be performed along the lines joining vertices in a network and that these actions would be called connections and second, the vertices of the problem solving schemas would contain states of knowledge of particular instances of underlying concepts.

With these principles established, the model development process unfolded as follows: (1) problems involving the definitions were chosen, (2) methods of solving these problems were examined, (3) states of knowledge utilized in problem solutions were identified, (4) concepts were identified for which the states of knowledge were about instances of those concepts, (5) problem solving methods were converted into problem solving paths which involved performing actions (connections), each of which begins with a state of knowledge of an instance of one of the identified concepts and ends with a state of knowledge of an instance of another (possibly the same) concept, (6) a figure (parent schema) was drawn in which each vertex represented one of the identified concepts and in which two concepts were directly joined by a line (link) if and only if, in at least one of the problem solving paths, states of knowledge of instances of the two concepts were directly joined by some action (connection), and (7) a second figure (problem solving schema) was drawn in which the vertices have the same geometric relationships to each other as the corresponding vertices in the parent schema have to each other, but in which two vertices may be joined by more than one line, each of which represents an action (connection) and is labeled. The result of this work, the preliminary specific model, has four sections: the right triangle interpretation (R-schema), the coordinate system interpretation (C-schema), the unit circle interpretation (U-schema), and the T-schema which represents a unification of the C- and U-schemas by examining some relationships between them. The R-schemas are sketched below.



PARENT SCHEMA


THEOREM OF PYTHAGORAS
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OBSERVE
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CONSTRUCT
CALCULATOR
TABLE
SOLVE
L
R
FORM RATIO
UNFORM RATIO
TAKE RECIPROCAL

PROBLEM SOLVING SCHEMA

Similar sketches were created for the other schemas to complete the preliminary model.

MODEL REFINEMENT

Four subjects were chosen from a college trigonometry class --an A, a B, a C, and a D student. Each was interviewed four times, once for each schema. Although some interview instructions were developed prior to the beginning of the interviews, they were modified and expanded during and after analysis of the interviews held with the first two subjects. The interviews followed this pattern: (1) pose a problem,

(2) observe the response and ask the subject to describe the method used, (3) exclude that method from possible use, (4) repeat steps one through three until the subject either follows each appropriate path from the model or demonstrates an inability to do so. On occasion a subject would follow a non-model path which caused a connection never to be reached or used. When this happened, conditions were set up, within the context of the problem, which allowed determination of the subjects ability to use that connection.

Each interview was audiotaped and was analyzed in such a way so as to produce a problem solving schema for each subject which represented the connections each subject used. From such a picture a parent schema for each subject for each of the model schemas was developed. Such a parent schema had represented both the concepts (as vertices) and the associations among them (as links) much like those of the model. Exactly what constituted the associations was determined by examining the connections between the two vertices in the corresponding problem solving level.

After analysis of all the interviews with the first four subjects, only slight refinements in the model were found to be needed. No changes in the R-schema were necessary, and the modifications in the other parts were very minor.

RELIABILITY AND VALIDITY

Two additional subjects were chosen from a different class, one an A student and the other a D student. They were interviewed using the refined model and the more precisely defined interview outline developed with the first four subjects. The results were analyzed and then re-analyzed by a second evaluator following the analysis guidelines developed.

These interviews were successfully completed and their analysis allowed the creation of a description of the subjects' schemas as a network of linked vertices that was consistent with the refined model. The agreement with the second evaluator was extremely high for the R-schema, slightly less so for the C-schema, and differed with some significance for the U-schema. Word association tasks completed by the last two subjects were consistent with the model. In addition, the schemas of the A student were significantly more complete than those of the D student.

CONCLUSIONS

The study developed a methodology that determined the conceptual structures of some students of trigonometry. By an analysis of problem solving behavior, links between concepts were inferred. The resulting model of trigonometry (as it is in the minds of students) has potential for use in the design and evaluation of instruction. More generally, the methodology can be applied to other domains in mathematics.

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SELECTED PROBLEM-SOLVING PROCESS AND PRODUCT VARIABLES IN THE SOLUTION OF MATHEMATICS WORD PROBLEMS

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Three studies examine the relationship of heuristics, cognitive style, and information selection to the number of correct solutions in mathematics word problem tasks. In the first study, 31 secondary students, enrolled in a summer Georgia Honors program, used a retrospective self report instrument to check the heuristics employed in the solution of each word problem. The best problem solvers, based on the number of correct solutions, were those students who reported using the greatest variety of heuristics both within and across problems. Each of 38 eighth grade students was observed in a clinical setting for two problem sessions in the second study. The correlation between conceptual tempo latency and the latency to first selection of information was significantly positive as was the correlation of field independence and product scores. Problems in geometry were the tasks in the third study with 92 secondary students who had completed a geometry course as subjects. There were no significant correlations found between field independence and the number of drawings selected or between field independence and product scores.

The National Council of Teachers of Mathematics booklet, An Agenda for Action, (NCTM, 1980) contains recommendations for school mathematics of the 1980s. The first recommendation is that problem solving be the focus of school mathematics. In recent years, mathematics education researchers, (Kantowski, 1974, 1978; Kilpatrick, 1969; Webb, 1975) and many others, have investigated various aspects of the processes involved in the solution of mathematics problems. Such research in problem solving is needed to give an empirical foundation to the selection of appropriate instructional methodologies related to the teaching of problem solving in the mathematics classroom. Selected results of three research studies involving the further examination of process and product variables in problems solving are described in brief. Some possible directions for future related research are posed as an outgrowth of these studies.

HEURISTICS USED BY PROBLEM SOLVERS

The question of how good problem solvers solve mathematical word problems led to the investigator's first attempt to trace a student's problem-solving process behavior. The purpose of the study was to characterize good and poor problem solvers (measured by product scores) in terms of the number of techniques they used in a solution. To obtain a record of those techniques used by students to solve a set of word problems a retrospective self-report methodology was employed.

The subjects were 24 volunteers from a group of 80 mathematically talented high school students enrolled in a Georgia Governor's Honors Program. These students were rising juniors and seniors selected through a statewide competition. Private and public secondary students from 17 different schools comprised the sample.

The investigator devised a self-report written checklist technique. This retrospective technique employed an 8 x 8 technique-by-problem matrix. The mathematical word problems in Webb's (1975) Problem-Solving Inventory were used. The techniques listed included categories such as use of an equation, recall of a theorem, and use of a drawing. The students were given a brief introduction to the meanings of the categories of techniques and the use of the checklist matrix. A maximum of 40 minutes was allowed for the solution of the problems. The students were atypical of the population of high school mathematics students and thus found several of these "problems" to be "exercises". Their retrospective accounts produced results that were hardly surprising: the best problem solvers (based on product scores) were those students who reported using the greatest variety of techniques both within and across problems. This finding is in accord with that of Webb (1975) who reported that students who used a wide range of heuristic strategies, on the average, were better problem solvers.

COGNITIVE STYLE AND INFORMATION SELECTION

Information students selected in solving mathematics word problems and the relation of information selection to two cognitive style variables, conceptual tempo and field dependence/independence, were investigated in the second study. Mathematics word problems about geometric figures were used. The study was designed to answer the following questions about students' mathematics problem solving: (1) What are students' preferences in the selection of supplemental information? (2) What is the relationship of conceptual tempo to information selection and problem solving? (3) What is the relationship of field dependence/independence to information selection and problem solving?

Cognitive style refers to the characteristic self-consistent way an individual has of functioning in the cognitive sphere (Witkin, 1969). It was hypothesized in this study that a student's cognitive style influences the selection of supplemental information in solving mathematical word problems. A tendency to field dependence or field independence, respectively, is a tendency to view a situation in its entirety, that is, globally, or to view a situation in its component parts, that is, analytically. This individual differences dimension, first identified in the field of perception, "shows itself equally in the problem solving domain," an area of cognition (Witkin, Moore, Goodenough, & Cox, 1977, p. 8).

Conceptual tempo refers to a person's consistent tendency to respond quickly or slowly in problem situations. Thus, the student who tends to respond quickly in difficult problem situations is categorized as impulsive, and the student who tends to reflect upon alternatives and to consider the quality of the product before responding is categorized as reflective. The concept of conceptual tempo was introduced by Kagan (1965) as a means of characterizing that part of students' problem-solving ability not accounted for by intelligence or verbal ability.

The 38 eighth grade mathematics students were categorized by cognitive style according to their scores on two tests: the Matching Familiar Figures (MFF) test for conceptual tempo and the Group Embedded Figures test (GEFT) for field dependence/independence. A Supplemental Information Selection (SIS) instrument devised by the investigator for each mathematics problem provided students with four categories of information (hints, drawings, definitions, and facts and formulas) for selection. The dependent variables were time to first selection, category of first selection, number of selections, and product score based on the number of correct solutions. Each student was observed in a clinical setting for two problem sessions with three word problems about geometric figures in each session. Records of the time and sequence of students' information selections during the solution process were used to describe their preferences in category of first selection and number of selections. Although the information selection varied from problem to problem, certain group tendencies were observed. There was a student preference for drawings as a first selection and for drawings and hints in the total number of selections.

The correlation between the MFF latency score and the latency to first selection of information was significantly positive ($p<.005$). The correlations of MFF latency score with the number of selections and the time spent on the problems were not significantly different from zero. A chi-square test of significance failed to support the hypothesis that more reflective students than impulsive students (classified by a double median split on MFF latency and error scores) are good problem solvers in terms of product scores. The positive correlation between GEFT score and product score ($p<.005$) was the only correlation between GEFT score and the dependent variables that was significantly different from zero. A chi-square test of significance supported the hypothesis that more field independent than field dependent students (median split on GEFT score) were good problem solvers in terms of product scores.

Both field independent and field dependent students preferred a drawing as a first selection. However, the field independent students preferred a marked drawing, the more complex of the two kinds of drawings, whereas the field dependent students did not exhibit this preference. The field independent students selected less information than their field dependent counterparts.

FIELD INDEPENDENCE AND INFORMATION SELECTION IN GEOMETRY

The third study was a collaborative effort to study processes that could account for differences in problem-solving ability in geometry among students who had achieved a mastery of the prerequisite knowledge base for the solution of nonroutine problems. The study, "Successful Problem Solving in Geometry and Selected Process Variables," was funded (1978-82) by the National Institute of Education under the direction of Kantowski at the University of Florida (Kantowski, 1978). The work of Polya and the teaching experiment of Soviet researchers underlie the methodology of this clinical study. One aspect of the study for the first year was the examination of the possible relationship of field independence and the student's selection of a drawing from an available hints instrument. It was hypothesized that field dependent students would select more drawings from an available problem specific hints instrument than their field independent counterparts. The hints instrument was patterned after the supplemental information instrument of the second study (Ehr, 1979).

Each of 92 students had two problem sessions during which six nonroutine problems in geometry were the given tasks. The Group Embedded Figure test as a measure of field independence was administered prior to the second problems session. The recording of process variables as a student worked a problem included a record of the sequence in which the student selected specific supplemental information from the hints instrument. Some students opted to work one or more problems without the use of the hints instrument. The categories of hints available were definitions and theorems, diagram, marked diagram, and plan for a solution.

The correlation of field independence and the number of times a student selected a drawing or marked drawing as a first selection during the twelve problem tasks was not significantly different from zero ($r = -.19$). Also, the correlation of field independence and the total number of correct solutions was not significant ($r = .28$). These results differ from the results of the second study. The difference may be the result of the procedures for the selection of subjects in the two studies.

In the second study the median GEFT score for the 38 eighth grade students was 12 with a range of 17 out of a possible 18 points. In a college sample, a mean score of 12 for males ($n = 155$) and 10.8 for females ($n = 242$) is reported (Witkin, et al., 1971). However, in the third study, the distribution of the 92 secondary students' GEFT scores was extremely skewed to the right with a median score of 16 and a range of 12. There were no significant sex differences for the students in the geometry project. These students were better than average mathematics students since they were post-geometry students who were required to have a prerequisite knowledge base of geometric facts for selection as subjects.

The tasks used to categorize students as field dependent or field independent are spatial visualization tasks involving the disembedding of a simple configuration from a complex diagram in which it is to be found. It is possible that field independence is related to success in problem solving in geometry and to the selection of a drawing, but the skewed distribution of scores for this study did not allow differences to emerge. The following questions need to be researched: 1) If a geometry study were done with average or low ability students, would the distribution of field independence scores be more normally distributed? 2) If such a distribution existed, would field independence be related to success in solving geometry problems?

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PSYCHOLOGICAL FACTORS AFFECTING STUDENTS' PERFORMANCE ON GEOMETRY PROBLEMS

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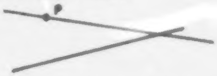
This paper examines, in detail, transcripts of college-level students working "out loud" as they solve straightedge and compass construction problems in plane geometry. We explore the role of "other than purely cognitive" factors as determinants of students' performance. The beliefs students hold about the nature of geometric argumentation and the role of mathematical proof "shape" their choice of cognitive resources and may determine how well they perform. Moreover, the nature of these students' belief systems -- despite their mathematical "sophistication" as veterans of the calculus, their approach to the geometry peoblems reflects a pre-Socratic empiricism — raises serious issues regarding the effectiveness of the curriculum. Students appear to learn just the opposite of what mathematicians would like them to learn, about the role of mathematical argumentation as a valuable means of thought.

The study of geometry can be justified on historical, intellectual, and aesthetic grounds. But perhaps the most compelling justification, at least so far as the mathematician is concerned, is that geometry is one of the rare microcosms in which students can become involved in doing honest-to-goodness mathematics. There is room for exploration and discovery, for generalization, for both inductive and deductive reasoning. In general, geometry supports the kind of reasoning processes typical of mathematical thinking at its best: the use of mathematical argumentation to test conjectures, to refine them, and ultimately to verify (i.e. prove) that they are correct.

I wish to stress here the importance of proving in its broad sense, in which one closes the gap between (current) conjectures and certainty by means of a dialectic that relies upon both intuition and rigorous mathematical reasoning [See Pólya (1954) and Lakatos (1976).]. The result of this dialectic is that the intuitions are tempered to the point where they are objectively verifiable, and a "proof" can be presented. Contrast this with the narrow sense, in which proving means presenting ironclad derivations of results that one already knows to be true. This latter category (verification through formal proof) is of minor general interest, even if one attributes to it the virtue of serving as a training ground for the development of "logic" and "precision" -- a virtue that, I must point out, did not succeed in maintaining Latin's position in the curriculum. But proving is much more. It is a means of understanding: the person who proves a result gets a sense of what "makes it tick" in much the same way that (for example) the person who assembles a clock comes to understand its inner workings. Even more important, proving is a means of coming to understand, and of coming to know what understanding is. In trying to prove something new, one is asking what makes it tick; in trying alternative proofs, rejecting them, modifying them, one is discovering things about its structure -- and solidifying one's knowledge in the process. This is the deep reason for much of the emphasis on proof in mathematics. The mathematician comes to accept proving as a way (if not the way) of thinking, a way of demanding and insuring that he does indeed understand.

Geometry is studied widely in our schools, and the vast majority of time devoted to it is generally spent in proof-related activities. What do our students learn about proving from all of that study? The college students who enrolled in my problem solving courses had studied geometry in secondary school, usually for a full year. They were competent at writing elementary geometric proofs. This study examined their beliefs about mathematical argumentation, as reflected by their behavior while they were solving geometry problems. Twenty-four students were videotaped, in pairs, as they worked on the following problem:

> You are given two intersecting straight lines and a point P marked on one of them, as in the figure below. Show how to construct, using straightedge and compass, a circle that is tangent to both lines and has the point P as its point of tangency to one of the lines.

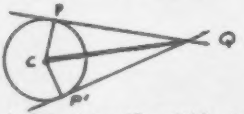


The problem can be solved by using any two of the following facts to locate the center, C, of the desired circle (see the figure below):

1. C lies on the perpendicular to the top line, through P.
2. C lies on the perpendicular to the bottom line through P', where P' is the same distance from Q as P.
3. C lies on the bisector of the angle at Q.

Fact (1) is a matter of memory, and verifying facts (2) and (3) is trivial. When they were given (2) and (3) as exercises after the videotaping sessions, e.g.

> Prove that if C is the center of the circle in the figure to the right, then the line CQ bisects the angle PQP',

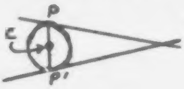


all twelve pairs of students dispatched them correctly within two to three minutes. Their actions while they worked on the problem, however, were quite something else. The mean time the students took to complete the problem (either to announce to me that they had solved it, or to run out of time at twenty minutes) was just shy of fifteen minutes. But what is striking about their behavior is the way that the students spent that time. The following is a synopsis of a representative protocol. The ways in which it is typical, and the issues raised by the approach the students take, are discussed below. Since space is limited, their discussion will be condensed, see Schoenfeld (in press) for some full protocols.

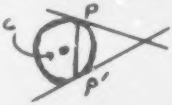
After reading the problem, students M and S draw some rough sketches and spend two to three minutes examining the features of them. Student M marks the perpendiculars at the points of tangency: he knows fact 1.

> S: how about this...if you put the thing [straightedge] here and draw something through P under it...you can make a straight line and bisect that.

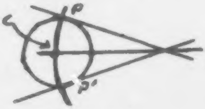
This is my students' most common conjecture (figure at left below), that the point C is the midpoint of the line segment from P to its "opposite," P'. The problem is done once one has C,of course. One of M's sketches (center) renders that implausible, and he proposes an alternative (right), in this conversation:



first conjecture



empirical rejection



alternate conjecture

> M: But you don't want a straight line, you want a curve.
>
> S: But once you have a straight line through those points, can't

you just...
M: Well, you can bisect the angle, right? Because you know the center point will be halfway...so you bisect this angle... then draw a curve, or do you? Hmm...wouldn't the center part be on an arc?

After a brief discussion they decide to test their conjecture by performing the indicated construction. They reach for a cheap compass that I have off to the side but within arm's reach, and spend about five minutes performing the construction. After a number of attempts they decide to reject their solution: the best they could do is represented in the figure to the left below, and they reject it visually as being inadequate. S suggests that they return to his earlier conjecture, and try to construct that. M is troubled about the point of tangency.

M: The point of tangency has to be the closest point, right?... Would that give you tangency at point P? Do you know what I mean?...there is a little bend here (where the arc passes through P), this is a right angle...umm, things don't look so good...but if we bisect...I mean if we draw a perpendicular through the line at P...

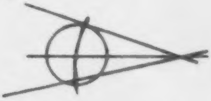
In the meantime, S has been examining their construction, and decides that it looks bad simply because they were clumsy:

S: That side [the top half of the angle] looks little wider than this [bottom half] side.
M: Yeah.
S: Why don't we try it again?
M: You try it that way and I'll try it the way I was thinking of... draw a perpendicular to the line at the point P...

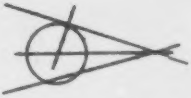
They ponder this last construction together, and decide it should work.

S: But can we prove that now?
M: Yeah, by drawing a circle and seeing if it works. (!)

Once again they pick up straightedge and compass, and perform the indicated construction. This time the construction happens to be correct. The compass they are using is inaccurate, however, and the construction that results (right, below) looks as bad as the one they had rejected previously.



first (false) conjecture,
rejected empirically



second (true) conjecture,
also flawed empirically

M: I think it is the same thing, this side looks bigger again. Yeah.
S: Definitely.
M: I think we've found it.
S: But we still haven't, really.

M: I think this way here (points to the second). This, I think.
S: Well, if we put them together...
M: There must be a surer way of doing it. [He is troubled about the first conjecture.]...That's the question...that's the thing about an arc, it is perpendicular.
S: Let's try it again.

They fiddle with the constructions, with the result that both come to look reasonable.

S: Now it looks just about right.
M: I think that's the thing about an arc, wherever it crosses another line,...because you are drawing a circle, actually... Like this, it is always perpendicular.
S: So we proved it two different ways.
M: Yeah, because it's tangent, so it works twice. All right, that's good.

They turn to me and report: "We can do it two different ways."

As strange as this tape may seem, the fact is that it is remarkably typical of the dozen that I recorded. The only significant difference between this tape and the others is that these students offered me two "solutions" to the problem -- and mutually contradictory ones, at that. However (see below), this reflects a difference in degree rather than in kind. The following characterizes the most salient features of this tape, and the degree to which that behavior was typical.

1. The students generate hypotheses through purely empirical and intuitive means. Potential solutions (loci to determine the point C) are chosen for implementation if they appear plausible in sketches made by the students. This occurred in all twelve tapes.

2. Verification is purely empirical: the way students test a hypothetical solution is to pick up straightedge and compass, and do the construction. If it looks right, they report success. If it does not, they either report failure or try another alternative (all twelve tapes). In the given tape we have the *reductio ad absurdum* of that empiricism. Two solutions, one of which is correct, meet the same empirical standards. Thus the two are accepted or rejected together.

3. More than 50% of the allotted time is spent with straightedge and compass in hand (Ten tapes of twelve; the others exceed 40%).

4. "Proof" is not mentioned, or if it is, is explicitly deemed empirical as in the remarkable quotation above: "But can we prove that now?"... "Yeah, by drawing a circle and seeing if it works." There was, in one

of the tapes, an after-the-fact substantiation of a construction: "We got it, that is, it looks pretty close...I think, if you center it right, they touch...OK. These are similar triangles....Yeah...They are equal triangles too." This was the only time in all of the problem sessions that any information was derived or confirmed by means of mathematical argumentation.

Before discussing the implications of these findings, I should point out that the twenty-four students whose performance is discussed here are (perhaps to your surprise) above average students. They were enrolled at a relatively selective institution, and had respectable College Board scores. Twenty-two of them had taken at least one semester of calculus (many had taken three), earning grades of A and B. Moreover, they had volunteered for my problem solving course -- which, given the course's reputation, was an indication of both motivation and confidence. The students came from a variety of geographical locations. All but three had taken a full year of geometry in high school. Thus whatever pathology these tapes reveal is not of local origin, but of global concern. Bluntly stated, that concern is as follows.

> Students come to believe that proving is a classroom game, an activity of little or no value outside the artificial classroom environment. This perspective becomes part of their belief system, (perhaps unconsciously) shaping their behavior. As a result, the students may not engage in any such activity on their own -- even when they have all the necessary tools at their disposal, and would profit greatly from using them.

I want to emphasize here that the students' mastery of proof techniques, in a classroom context, is not in question. It is not insignificant that each of the twelve pairs of students that was videotaped recalled fact 1 about the point C (see above) and was able, without difficulty, to derive facts 2 and 3 when I asked them to. They had learned their classroom lessons well. But that is precisely the point: in this slightly different context, that classroom knowledge went unused. I believe that most of the students took the assigned problem seriously, and wanted to solve it. (It does not make one feel good to be videotaped in failure.) This gives rise to two hypotheses. Either (1) the students do accept proving as a valid but distasteful form of argumentation, and invoke it only when the cost of not doing so is extremely high (higher than the cost of failure in this setting), or (2) they reject it altogether, as suggested above. In either case, this raises serious issues with regard to the curriculum. It points to a range of questions we must explore about students' perceptions regarding the role of mathematics and mathematical reasoning. How are such perceptions engendered? Once students hold them, how do they affect the mathematics that the students produce? And, if the effects are as negative as they seem to be: How can we structure the environment for students so that they learn the positive aspects of proving, and enjoy the benefits that can be obtained from studying geometry?

As this last question suggests, I wish to end on a positive note. I am firmly convinced of the value of mathematical argumentation, for all of the reasons outlined at the beginning of this article. I am also convinced that geometry is a superb microcosm in which to develop an appreciation of, and experience in, such mathematical argumentation. Creating instruction that will achieve this will be no easy task -- but it will be worth it.

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THE DEVELOPMENT OF PROJECTIVE SKILLS

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The ability to comprehend spatial arrangements from different viewpoints is discussed as it has been studied by factor analysts and developmental psychologists interested in spatial ability. Recent research on Piaget's three mountains task is used to suggest topics for research in mathematics education. Following Clement's (1981) call to develop spatial programs for children, with well-tested activities, the need to construct classroom activities that will utilize perspective or projective abilities is identified. Science education and Fuson's (1978) list of projective properties and relationships are suggested as sources of such activities, and some research topics are also suggested.

With the recent increase in interest in spatial ability, there have been new attempts to define it. McGee (1979) identified in the factor-analytic literature two spatial abilities, spatial visualization and spatial orientation. While the distinctions between these two abilities are not clear, it appears that spatial orientation includes the ability to comprehend spatial arrangements from different viewpoints.

Developmental psychologists working within a Piagetian framework have been concerned with the description of the progression through stages of a child's ability to think about space. According to Piaget (1956), a person who has achieved representational thought has mastered projective space. This entails the ability to consider an object or pattern in relation to a particular point of view. If the point of view is that of the person himself, then perspective relationships are involved. If the point of view is that of other objects on which the first object is projected, then projective relationships are involved (Piaget, 1956, p.153).

As Clements (1981) has pointed out, the work of the factor analysts and the developmental psychologists is complementary, although there is little communication between the two groups. Certainly, research on projective tasks should be of interest to both groups. Recent research on Piaget's (1956) original projective tasks may suggest some fruitful ideas for mathematics educators. Let us consider one such task as an example.

THE THREE MOUNTAINS TASK

Piaget's (1956) three mountains task explored the development of children's ability to coordinate perspectives. His results showed that until the age of about 9 or 10, children were unable completely to free themselves from egocentric responses that assigned their own viewpoints to a doll whose viewpoint they had to construct or select from photographs. Any theory of spatial ability needs to accommodate this phenomenon.

Subsequent research exploring the task variables in this problem has, however, shown that egocentrism may be partly an artifact of the problem itself. For example, Cox (1975) found fewer egocentric responses when the doll was replaced by a person, and Garner and Plant (1972) found that requiring children to select their own view first increased the frequency of egocentric responses. Another frequent result is that children as young as 3½ years are capable of nonegocentric responses, provided the task is made easy enough (Borke, 1975; Fishbein, Lewis, & Keiffer, 1972). Egocentrism may therefore also be a response to task difficulty rather than a characteristic of the spatial ability of children at a particular age.

Two lines of thought are prompted by these results. What particular features of a task requiring perspective skills make it easy or difficult for children, and how can these difficulties be overcome? Is a spatial ability like the ability to coordinate viewpoints a unitary construct? There have been many studies addressing the first of these questions. See for example the study by Fehr (1977), which used as variables the nature of the observer, the type of choice stimuli, and the presence or absence of a landmark.

The answer to the second question may be that perspective-taking is two separate achievements, the recognition that (a) one observer has one unique view, and (b) one view cannot be seen from more than one position, with (a) easier than (b) (Salatas & Flavell, 1976). If this is so, then classroom activities that take this fact into account, and help children to progress from (a) to (b) should be useful to them.

A different line of research is suggested by another study in this field. Huttenlocher and Presson (1973) contrasted the performance of children in a perspective problem and in a comparable rotation problem, with the second task proving easier for the children in their sample. The rotation abilities required in their task seem to be related to McGee's (1979) spatial visualization factor. Is the ability to imagine an array rotating while one is stationary different from the ability to imagine oneself moving around a stationary array? If it is, we could investigate the relationship between the two abilities in a mathematics education setting; in particular, which skill do children actually use in problems where both might lead to a successful solution?

NEEDED STUDIES IN MATHEMATICS EDUCATION

Clements (1981) has argued that our greatest need is for practical spatial teaching programs and their experimental testing, with special regard for their effectiveness in promoting retention and transfer. The best of these programs will offer a wide range of carefully sequenced tasks.

One example of a study of retention is that of Smith and Schroeder (1979, 1981) in which 4th-grade children who had been instructed for ten half-hour lessons on tangrams were retested a year later on a spatial visualization test. Instructed students performed better than a non-instructed control group. This kind of study appears not to have been undertaken with lessons requiring perspective or projective skills. This may be partly because mathematics educators have not yet given attention to what might constitute an appropriate sequence of lessons. Clearly, we need to devise classroom activities with more ecological validity than Piaget's three mountains task.

Science educators have recognised the importance of projective and perspective skills in various topics. Bishop (1978) gives examples of applications of these skills in earth sciences (including astronomy, geology, and meteorology), biology, and chemistry. Thier (1976) describes an activity in which children are led to identify the position of an object relative to a model observer, "Mr O," whose own view is that of the child at first, but who then gradually changes to new positions. Science curriculum projects covering the above topics or including activities like "Mr O" might suggest activities that could be used in mathematics education.

Fuson's (1978) list of properties and relationships that remain invariant under projective and affine transformations suggests that classroom activities that focus on collinearity, the number of sides of a polygon, convexity, betweenness, parallelism, and ratios of lengths along a line, might be particularly appropriate for developing projective and perspective abilities. Dienes and Golding (1967) suggested activities using this classification, and these have been particularly fruitful for the present writer in work with ten-year-old children. However, unless the use of perspective or projective abilities is explicitly required, then it is possible that these abilities will not be used at all in such lessons. There is an opportunity here for mathematics educators to explore how children approach such spatial tasks, which may contain a nontrivial mathematics component.

If appropriate classroom activities can be designed by mathematics educators, perhaps based on Fuson's (1978) list, and requiring explicit use of perspective or projective abilities, then some of the lines of research suggested by the developmental psychology literature could be pursued. For example, such activities might provide a vehicle for studying the development of spatial orientation in children, leading to a clearer description of this ability than we now have. We might also compare the use of perspective-taking skills and rotation skills in comparable mathematical settings, thus throwing light on the relationships between spatial visualization and spatial orientation, and providing insight into how spatial programs can be used to promote transfer of spatial skills.

Perspective and projective tasks could be described using the DIPT classification scheme designed by Wattanawaha (1977; cited in Clements. 1981). The DIPT scheme is a proposed classification of spatial tasks based on four independent characteristics: the dimension of thinking required by the task, the amount of internalization required, the manner in which the answer has to be presented, and the type of thinking required by the task, in particular whether the child needs to work out the mental operations to be used. Each characteristic has associated with it a number of values. For example, internalization has values of 0, 1, or 2 depending on whether the task requires no visual image to be formed, or requires a visual image without the need to transform it in the mind, or requires a visual image which must be transformed in the mind (Clements, 1981). A task with an internalization value of 2 should prove harder for children than a task with an internalization value of 0, provided the values on the remaining characteristics are the same.

Wattanawaha's (1977) scheme suggests a way that the levels of difficulty of projective and perspective tasks could be assessed, based on their demands on children's thinking. The scheme might also highlight where the search for classroom activities should be focussed. For example, if we find that there are many activities requiring internalization at levels 0 and 1 but few requiring internalization at level 2, then we might concentrate on constructing activities in this last category. Equipped with a range of perspective and projective tasks with well-defined levels of difficulty, and nontrivial mathematics content, we might be closer to devising the well-sequenced spatial programs Clements (1981) has called for.

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MODELING STUDENTS' MODELING BEHAVIORS

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This paper will describe some recent results from an NSF funded Applied Problem Solving project. It will focus on the general "modeling process" that two groups of students went through during two problem solving sessions. The characterization of problem solving that emerges from our protocols is quite different from information processing models that are becoming increasingly popular in mathematical problem solving research. Rather than producing answers by linking together stable structures (e.g., production systems), solutions were constructed by gradually organizing, integrating, and differentiating unstable structures (i.e., conceptual models). Furthermore, the most beneficial problem solving techniques were those focusing on conceptual rather than procedural considerations, and those serving to help the student overcome cognitive characteristics associated with unstable conceptual models rather than those intended to amplify cognitive capabilities.

The purpose of this paper is to describe and illustrate, by focusing on two video-taped problem solving sessions, the "modeling process" that characterized most of the sessions from our Applied Problem Solving research. The two problems selected for discussion here were used in a component of our project that consisted of a sixteen week problem solving "course" involving 23 average ability seventh graders. This component of our project has been described in Lesh (1981 a); detailed protocols of the two sessions are given in Lesh (Note 1), and interpreted transcripts of the sessions will be made available at the Georgia PME meeting. (Copies of these or other problems and transcripts can be obtained by writing to the author.)

During the problem solving course, students spent two days each week working in groups of three on problems that might reasonably have occurred in their everyday lives. In Lesh (1981 b), Lesh and Akerstrom (1981), and in Lesh, Landau, and Hamilton (note 2), we have described how our tasks differ from those that have dominated problem solving research in mathematics education; and detailed descriptions of some of our problems have been given.

Each of the 18 problems used in our group problem solving sessions was designed to require from 10 to 45 minutes for solution, and realistic "outside" resources were available (including other students, calculators, resource books and materials, and teacher consultants who also acted as observers). All of the problems were based on straightforward uses of easy-to-identify concepts from arithmetic, measurement, or intuitive geometry; no "tricks" were needed - the most direct path was a correct path (although a variety of solutions and multiple-step solution paths were usually possible).

Our problems were not the type in which students needed to get from A to B within a well defined problem space. The definition of problems, givens, and goals all required input from the student. For most of the problems, non-answer-giving stages (such as problem formulation, trial solution evaluation, etc.) were critical, and solutions could vary in both type and sophistication. The goal was not simply to arrive at a mathematical answer; mathematics was a means to an end, not an end in itself. For example, a non-mathematical decision might have been made, using mathematics as a tool.

Six different groups of students worked on each problem, and each session was closely observed, video- or audio-taped, transcribed, and interpreted by several members of our research team. Then, transcripts, observations, and interpretations were analyzed in a sequence of project meetings involving participants representing a variety of theoretical perspectives: mathematics, mathematics education, developmental psychology, social psychology, etc. Project discussions focused on modeling, the use of representations, data management, critical elements (i.e., specific mathematical skills, processes, and understandings), group dynamics, metacognition, individual differences, task characteristics, etc. In all, 117 sessions were audio- or video-taped, transcribed, analyzed, and interpreted.

The main point that I want to emphasize here is that, from a global perspective, the most appropriate way to charcterize most of our sessions was that the group (or individual) alternated among several distinct interpretations or models of the situation (including goals and givens), with different conceptualizations usually occurring at various degrees of refinement and sophistication, and the early models being quite unstable.

If problem solving is characterized as a process of linking together stable procedural systems (e.g., production systems from information processing theories), then the major theoretical issue is to explain how these systems are selected and sequenced, i.e., to explain the conditions under which they are activated. In contrast, we found that solutions to our problems typically were constructed by refining, integrating, and differentiating initially unstable conceptual systems (or models). Thus, the major theoretical issues that our research is designed to address are: (a) to identify the nature of students' primitive conceptual models for various mathematical ideas, (b) to explain how deficiencies in various models are detected, (c) to explain how successively more refined models are gradually constructed, and (d) to describe cognitive characteristics associated with the use of unstable conceptual models, and how their counterproductive influences can be minimized. These issues have been addressed in past publications from our project (e.g., Lesh, Landau, and Hamilton, Note 2), and they will be the foci of many future publications; there is not space to address them adequately here. The goal of this paper is to deal with a more global issue, that is, the most natural way to characterize (and ultimately model) the general modeling/problem solving procedures that students used to solve our Problems.

THE PROBLEMS:

In the first problem that I want to discuss, the students were asked to "pretend that you are a counselor at a sports camp." Their job was to sort twelve incoming campers into three groups using any or all of the following information which was available for each camper: 50 yd. dash times, 100 yd. dash times, high jump distances, long jump distames swimming levels, and coaches' recommendations.

In the second problem, the group was taken to a simple 19' x 19' room, handed a roll of wallpaper (with a label giving the length, width, and area of the roll) given yardsticks, calculators, tape measures, etc., and presented the following problem:

> You have a roll of wallpaper. Each roll costs $17.35. The wallpaper store does not sell parts of rolls and they will only return half the cost of unused rolls. Determine how much it will cost to wallpaper this room.

THE RESULT:

In order for my major point to be understood, it is important to clarify what I mean by "alternating among several interpretations or models of the problem." However, space limitations in this paper preclude the kind of detail that is needed for full discussion of my topic. Therefore, I will give extremely brief descriptions of the models and modeling processes that are apparent in our protocols, and refer the reader to other project publications for further explanations, illustrations, and data.

In the "camp placement" problem, our students interpreted, organized, and manipulated the problem in four distinct ways: (a) build up a rank order list, based on successive two-term comparisons and using global or shifting criteria, then divide the list into groups; (b) sort out groups, based on global or shifting criteria; (c) combine the data into a single "score" for each camper, using an all encompassing equation/formula/rule (e.g., average, sum), then divide the list into groups; and (d) rank (or group) the campers within each event separately, then find a way to combine rankings (or groupings) using a procedure that is either stochastic or non-stochastic. The non-stochastic interpretation assumed from the start that there was a simple, powerful formula for combining information. The stochastic interpretation formed a trial answer based on only a subset of the given information, then refined a sequence of progressively more sophisticated answers each of which accounted for successively more information.

In the wallpaper problem two basic interpretations were used. One interpretation treated the problem as a "find the area of the room" problem, where area was interpreted as "length times height" and the entire area of the wallpaper was considered to be usable. The second interpretation was more direct and concrete, and consisted of finding how many wallpaper strips (interpreted as "pieces of area") it would take to cover the wall. In general, the first interpretation was more "mathematical" from the perspective of the students, but the second tends to yield better practical results.

Most groups, at some time in their solution attempts, showed signs (perhaps primitive) of thinking about the problem using variations on each of the basic interpretations. Successive conceptualizations of the problem filtered, organized, interpreted, and manipulated "given" information in different ways; then these interpretations led to more refined conceptualizations, which in turn led to different ways to select, organize, and interpret the goals and givens. Progressively more sophisticated models took into account more information, and imposed fewer irrelevant or debilitating restrictions or subjective false assumptions. For the early, less stable models, half-formulated problem conceptualizations suggested half-formulated solution strategies and unarticulated (and unexamined) alternative ways to filter and distort information and data. When a group initiated the first, second, or third procedure associated with a given conceptualization, they frequently lost cognizance of the overall conceptualization or solution process, and began to implement a procedure associated with an entirely different conceptualization. Then, the revised problem interpretation might suggest a completely new solution process. Sometimes, this sort of "mutation" led to the generation of productive ideas and problem interpretations; in other cases, it derailed promising solution efforts.

During the solution attempts to our problems, the most beneficial decisions that our students made were seldom (solely or even primarily) of the type, What shall I do next?" In fact, students and groups who seemed preoccupied with doing in their planning, monitoring, and assessing activities typically did not do well compared to their peers. Groups which succeeded spent relatively more time thinking about ways to think about the problem (both goals and givens) rather than ways to do it, or get from givens to goals. Conceptual considerations seemed to be far more productive than procedural considerations, especially if the procedural considerations were of the general or content-independent variety emphasized by, for example, Polya (1957).

All of the above observations will be apparent in the interpreted transcripts that will be presented at the PME meeting.

CONCLUSIONS

Our research has not attempted to compare "experts" with "novices" in the problem domains we have investigated. Instead, our comparisons have been between "novices who succeed" and "novices who fail" on given problems. By our definition, experts in a given problem domain are people who have acquired stable and accessible conceptual models for interpreting and manipulating information in that area. Therefore, even when these individuals do not have an immediate response to a question, they do have well organized conceptual and procedural systems for producing a response. Thus, according to our restricted definition (see Lesh, 1981a or Lesh, Note 1), the situation is not really a problem - it is just an exercise.

For us, a problem is a situation in which a stable conceptual model is not available to the student (or group). So, the problem cannot be solved simply by linking together stable systems (e.g., production systems). If information processing were going on in our sessions, the information was not simply raw data, or even multiple attributes associated with raw data. Usually, it involved organizational/relational systems that had to be imposed on the data using the students' conceptual models. It was (primarily) these organizational systems and systems of relationships (i.e., the conceptual models themselves) that were being "processed" during solution attempts, not the given information; and the processes that facilitated these modeling activities were the type that differentiated successful from unsuccessful groups. The primary factors that seemed to influence cognitive functioning in mathematical learning and problem solving situations were related to the stability of students' underlying conceptual models, and the processes that were most helpful were those which enabled them to cope with cognitive characteristics associated with unstable conceptual models. (For a description of conceptual models, our basic theoretical construct, the reader must be referred to other publications from our project, e.g., Lesh, Note 1 or Lesh, Landau, and Hamilton, Note 2.)

Whether students use a "modify a single all-inclusive idea" approach to problem solving, or a "piece together several simpler ideas and procedures" approach, both tend to involve modifying underlying models (i.e., ways to think about the problem) to make them fit the problem situation. This "tailoring" that is required to make a basic model fit a real situation is one of the most important components of the "modeling" process that we believe lies at the heart of (even elementary) applied problem solving. Unfortunately, many of our students had learned from their school experiences that this sort of "tinkering" is equivalent to "not knowing the answer," and is undesirable in mathematics.

We believe that the model of problem solving that is evolving from our research is quite different from existing popular theoretical perspectives, and that its instructional implications are direct and potentially very effective for improving the abilities of average ability students to use mathematics in everyday situations.

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APPLIED MATHEMATICAL PROBLEM SOLVING:
RESULTS FROM ONE PROBLEM FOR SEVENTH-GRADERS

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A major component of the current NSF-funded project on applied problem solving was the observation of groups of three seventh-graders working together on a single, large problem for an entire class period, approximately 35 minutes. This paper discusses one problem (in as much detail as space permits), comparing the solutions of several groups of subjects. The problem is one in which quantitative judgments must be based on qualitative information and for which no "right answer" exists. Subjects must decide how two people who bring different abilities and assets to a profitable business situation should divide the resulting income.

This paper for the conference proceedings will be supplemented at the presentation with a videotape portion of one of the sessions, transcript portions of the other sessions, and suggestions for improving the problem for future use.

The purpose of this presentation is to discuss one of the problems used in the Applied Problem Solving (APS) Project, an NSF-funded research project at Northwestern University. This report will briefly outline the APS Project, state and describe characteristics of one of the problems used in the project (the "computer problem"), and describe solutions constructed by our subjects.

The chief goal of the APS Project is to identify the most important processes, skills, and understandings that are needed by average ability middle school students when they attempt to use substantive elementary mathematical content to solve realistic problems that might occur in the everyday lives of students or their families. More information on the design of the APS Project can be found in Lesh, Landau, and Hamilton (in press).

A whole seventh grade class of 23 average ability students participated in a sixteen week problem solving course during the fall semester of the

1981-82 school year. Two days per week were devoted to taped sessions in which groups of three subjects worked for an entire class period (about 35 minutes) on realistic, applied problems.

Each of the 18 problems used in the groups sessions was designed to require from 10 to 45 minutes for solution. Realistic "outside" resources were available, including other students, calculators, resource books and materials, and teacher/consultants who also acted as observers. All of the problems used easy-to-identify concepts from arithmetic, measurement, or intuitive geometry. For most of the problems, such non-answer-giving stages as problem formulation and trial solution evaluation were critical. Often the goal was not simply to arrive at a mathematical answer, but rather to make a non-mathematical decision or set of decisions using mathematics as a tool. In other cases a mathematical decision based on both quantitative and qualitative information was required. This was a central characteristic of the computer problem:

> Bill bought a small computer to use with his job. He spent about $5000 for the computer. Bill spent a long time deciding which computer to buy, and he spent an even longer time learning how to use it. But after a few months, he found that he needed it only a few hours a week.
>
> His friend Jon didn't know how to use a computer, but he found a business willing to pay $240 per month for a certain computer job. This is what the job involved:
>
>> Job A: Bill had to write a computer "program" to tell the computer how to do the job. This took about eight hours, and required Bill's special knowledge about computers. Once the program was written, it could be used again and again each month after that. Hence, Bill did not do any work after the first month.
>>
>> Jon had to "feed" the computer information that the computer used to do its job. This took eight hours a month, and it was work that anyone could do. Jon did this each month.
>
> Jon and Bill could not agree on how to share the money. Bill argued that he owned the computer and had spent a lot of time learning to do something Jon couldn't do. Jon argued that he found the job and did all the work after the first month.
>
> Even before they settled the argument on how to split the money, Jon was able to locate three jobs from other businesses. This is how much each of the other three jobs pays each month, and how much time is required to do them:
>
>> Job 1: Pays $70 each month. Bill writes one program

(that takes 4 hours) and Jon feeds the computer each month (that takes 10 hours a month).

Job 2: Pays $200 a month. Bill writes one program (that takes him 20 hours) and Jon feeds the computer each month (that takes 7 hours a month).

Job 3: Pays $95 a month. Bill writes one program (that takes him 8 hours the first month and 1 hour each month after that) and Jon feeds the computer each month (that takes 9 hours a month).

Decide how Jon and Bill should split up the money each month for each job. (First decide how they should share the money for Job A, then for Jobs 1, 2, and 3.)

PROBLEM CHARACTERISTICS

Problem characteristics are underlined in the discussion below. As stated, the central characteristic is that this problem required quantitative conclusions based on both qualitative and quantitative conditions. The quantitative information to be incorporated in the solution included the amount of time Jon and Bill each put into the jobs, the income from each job, the numbers the problem solvers generated by computing percentages, hourly wage possibilities, etc., and, for most groups, the tentative duration they assigned to the jobs. (Job duration information was intentionally omitted from the problem statement.)

The following elements favor extra compensation for Bill: he is the sole owner of the computer; he must program the computer, whereas Jon only has to keypunch data; and he made a sizable time investment in purchasing the computer and learning how to use it. These qualitative elements must be weighed against the facts that Jon found the jobs, that Jon does all of the work after the first month (except in Job 3), and that without the jobs the computer is of limited value to Bill.

No single group fully assimilated and processed all of this qualitative information. Because there was no right answer, and because the excessive information was qualitative, this problem was amenable to reasonable solutions at several levels of sophistication.

Another distinguishing feature of this problem is that it required the students to generate a succession of models, each of which was useful but inadequate for solving the next set of problem conditions.

Finally, the problem was designed to be realistic. Students became very involved in this problem. Despite its length, the students did not have any difficulty understanding the problem.

SOLUTIONS

The solutions produced by four groups solving this problem are outlined in order of increasing sophistication.

Group A. The weakest solution initially incorporated no qualitative information. Eventually, the students were trying to coordinate three pieces of information: Bill's ownership, his programming expertise, and the different number of hours put in by each worker. The group decided to have Jon and Bill share the proceeds from Job A equally, and for Jon to receive $50 and $110, respectively, from Jobs 1 and 2. They did not get to Job 3. Their attempt in the middle of the session to incorporate ownership failed; after they brought it up they forgot to include it in their final formulation of income shares. They did not vary income shares across months.

Group B. Group dynamics and solution were hindered by an argument between one student, contending that Jon should receive a compensation equal to that of the owner, and another student, adamant that Bill's ownership of the computer entitles him to compensation even during the months when he does not work. The group did not initially notice that Bill did not work after the first month, so they decided that Bill and Jon should both be paid (equal) hourly wages. After they tried to implement the same model on Job 1, the observer suggested that they return to the first job and consider the facts that Bill did not work after the first month and that the jobs were open-ended. This led the group to specify the duration of the jobs as one year, and to distribute the job income equally after the first month. Before they reached that point, however, they had the heated discussion mentioned above. They did not use the information that Bill wrote the programs that were used each month.

Group C. This group developed a fairly sophisticated solution. The most powerful contribution was made by the one girl in the group who took a comprehensive view of the difficulty of dealing with the qualitative information and perceived the expedience of making quick and arbitrary (though reasonable) decisions. This group did not incorporate Jon's finding the jobs, although ownership and programming expertise were considered. The boys quickly agreed when it was suggested that the shares for the first month should be different from the shares for subsequent months. This group solved for three of the four jobs, and decided that for the first month Bill would be paid $435 and Jon $75; the second month the shares would be $125 for Bill and $385 for Jon; and for each month after that, $110 for Bill and $400 for Jon.

Group D. This group had the most comprehensive and complex solution. All of the qualitative elements were introduced into the solution, though not to the same extent. As in the other groups, the most important qualitative factors were ownership and programming expertise. Unlike other groups, this one more successfully integrated factors favoring Jon's compensation. Additionally, they generated and integrated into their solutions a sensible hourly wage figure for keypunching. Like Group C, they varied income shares across months. This solution was more sophisticated because (a) its construction coordinated more qualitative and quantitative information; (b) it included all four jobs; and (c) it involved different shares over four months (with the shares remaining the same after the fourth month), whereas Group C's solution, with one exception, only differentiated between the first month and the following months.

DISCUSSION

Tapes of the four groups were critically examined to determine why the solutions varied so widely in sophistication. Academic ability, involvement in the problem, and the presence of at least one "real world expert" (an individual able to transmit a more comprehensive grasp of the problem to the other group members) were similar across groups and so were discounted as principal factors.

Several elements uniquely characterized Group D, and are suggested as factors critical to its success. Their "real world expert" not only contributed a more comprehensive grasp of the problem to the group, but, along with the other two group members, established the goal of achieving consensus in weighing the qualitative and quantitative information. This commitment to consensus affirmed the value of each member's opinions and so created an atmosphere in which ideas were more freely expressed. Such an atmosphere was important for a problem with as many "fuzzy" factors as this one. The open environment elicited more of the intuitive thinking required for coordinating such "fuzzy" factors; because the intuitions were usually accepted as they were introduced, consensus and cooperation proved viable and developed into a momentum that saw the session through to a very successful conclusion.

Another critical factor was an observer intervention about half way through the session. The group weighed so many different factors that they could not agree to a fair division of income. The observer suggested that they could split the money differently each month. This idea seemed to break up a log jam in the session and was clearly a key to the group reaching such a sophisticated solution before the session ended. In the absence of the excellent group dynamics, however, the suggestion would not have been so useful. The idea to vary income shares across months occurred early in Group C's session. Group dynamics in that session were mediocre, however, and revolved around one person; though she was bright (brighter than the members of Group D), and though quite capable of making quantitative judgments based on qualitative data, she was unable to lead her group to a solution as complex or comprehensive as that constructed by Group D.

For all of the groups, competence with simple pre-algebra concepts (rates, percents, addition, etc.) were required givens. All of the groups had access to calculators which were used sporadically between computations with paper and pencil.

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STRATEGY USE AND EFFECTIVENESS IN SUCCESSIVE CONCEPT ACQUISITION TASKS[*]

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Three experimental groups of female college undergraduates (N=108) performed two successive concept acquisition tasks, which varied in the diversity of objects presented and in the perceptual immediacy of the relevant attributes. New scores for "intended focusing" and "intended scanning" strategies were defined based on structured questions posed during each interview, and compared with previously defined behavioral measures of strategy use. The effects of task and treatment variables on problem-solving success, strategy usage, and strategy effectiveness included the following: (a) experience with a more difficult task first improved performance on the second task; (b) intended strategy use and behaviorally measured strategy use were affected similarly by the task variables, but were uncorrelated with each other; (c) intended strategy use was considerably more influenced by prior experience on a related problem than was behaviorally measured strategy use; (d) neither intended focusing nor intended scanning was significantly more effective on any task, while a high behavioral focusing index was associated with problem-solving success. The findings suggest a reinterpretation of the extensive literature on concept acquisition tasks, and have implications for the way in which strategy usage is measured in problem-solving research.

INTRODUCTION

A conjunctive concept is a way of grouping things in terms of one or more shared characteristics (attribute values). Conjunctive concept attainment tasks require a subject to identify a conjunctive concept known to the experimenter. In the basic selection paradigm, a card depicting an example of the goal concept is shown to the subject who then chooses additional cards one by one to test, being informed after each choice whether or not the card selected is an example of the concept.

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[*] *This report is based on a doctoral dissertation by William Waters, 1979, at the Univ. of Pennsylvania, under the direction of Gerald A. Goldin.*

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The subject may also make a hypothesis as to the identity of the goal concept after any (or all) card choices. The task ends when the goal concept is correctly identified. Variations on the basic task include presenting the cards in a fixed sequence (reception paradigm), presenting no instance or a negative instance of the concept initially, informing the subject of the number of attributes relevant to the goal concept, etc.

Four main ideal strategies which subjects seem to exhibit in performing such tasks are: *simultaneous scanning* (simultaneously considering all concepts possible at each stage, and making the card choice which logically eliminates the largest number), *successive scanning* (observing one set of attribute values common to several positive instances of the concept, and hypothesizing that they define the concept), *conservative focusing* (holding constant the values of all but one of the attributes of a single positive instance of the concept, and varying the one attribute to determine if it is a defining attribute), and *focus gambling* (varying two or more of the attributes of a single positive instance while the rest are held constant, in the hope of eliminating them as defining attributes in a single trial). Studies of conjunctive concept attainment have indicated that focusing strategies are more efficient than scanning strategies (Bruner, Goodnow, and Austin, 1956).

Recently attention has been devoted to examining more systematically how the characteristics of problem tasks (syntax, content, context, structure, and heuristic behavior variables) affect problem-solving outcomes (Goldin and McClintock, 1979), and still more fundamentally to determining how such outcomes are best measured (Goldin, 1982). The present study investigates two main conjectures: (a) that the previously observed superiority of focusing strategies may result principally from task characteristics (the perceptual immediacy of the attributes and the limited universe of objects) which do not carry over into real-life situations; and (b) that the superiority of focusing strategies may be a consequence of behavioral strategy scoring systems, based on card choices and hypotheses, which tend to define non-redundant choices as "focus" choices, and redundant choices as "scans." To carry out the investigation new card tasks were constructed, and new strategy scores defined (Waters, 1979a,b).

CONSTRUCTION OF THE TASKS

Task 1, essentially a replication of Bruner's basic task, employs four perceptual attributes *shape* (square, circle, or triangle), *color* (red, green, or brown), *size* (large or small), and *number* (one object or two objects per card). In Task 2, the dimension *shape* is replaced by the less perceptually immediate attribute *animal life-plant life-inanimate object*. There is also a greater diversity of objects--the objects now vary over this attribute and the attribute of color. Thus the Task 1 cards "one small red circle" and "one small green circle" correspond in Task 2 to the cards "one small red apple" and "one small green cactus" respectively. Task 3 employs the same attributes as Task 2, but the objects vary over all four relevant dimensions. In each task the subject is presented with a board containing all of the cards "matrixed" across the attributes (Fig. 1); Waters (1979a,b) provides more detail, as well as the directions given to subjects prior to task administration.

Figure 1. *Task 2 (Schematic), Conjunctive Concept Acquisition*

1 S R	1 S R	1 S R	1 L R	1 L R	1 L R	2 S R	2 S R	(etc.)
apple	bell	fox	apple	bell	fox	apples	bells	.....
1 S G	1 S G	1 S G	1 L G	1 L G	1 L G	2 S G	2 S R	.....
snake	cactus	bottle	snake	cactus	bottle	snakes	cacti	.....
1 S B	1 S B	1 S B	1 L B	1 L B	1 L B	2 S B	2 S B	.....
clock	bear	tree	clock	bear	tree	clocks	bears	.....

Key: S = small, L = large; R = red, G = green, B = brown. On the board presented to subjects are *pictures* of the objects described, together with the printed word "apple," "snakes," etc. Note that the dimension *animal life-plant life-inanimate object* is matrixed diagonally.

The three tasks are as far as possible structurally isomorphic. Cards exemplifying corresponding combinations of attribute values occupy the same relative positions on the boards, while the goal concepts and the start cards are likewise assigned isomorphically.

STRATEGY SCORES

Since 1956, several authors have pointed out that it may not be possible to distinguish "focusers" from "scanners" solely from the sequences of card choices and hypotheses (Eifermann, 1965a,b; Wason, 1960; Wetherick, 1969). Often the same choice could be interpreted logically as either a conservative focus or a simultaneous scan, for example. Eifermann (1965a,b) and Giambra (1971) anticipate our present procedure by proposing that the subject state the reason for each card selected, as a way of resolving the logical ambiguities in strategy scoring. Laughlin, on the other hand, states:

> "... while recognizing the cogency of this argument, in my own work I have preferred to infer strategies strictly from the pattern of instance selections and accompanying verbal hypotheses, because of the possibility of interference with the thought processes by the verbal statements that has been well-demonstrated in problem-solving research, and because of the high correlations between actual behavior and the stated reasons for the behavior demonstrated by Schwartz ... and Gambria ..." (Laughlin, 1973, p. 281)

Laughlin and his associates vary their scoring rules for different tasks, requiring generally for focus choices that (1) nonredundant information be obtained on at least one attribute, and (2) the subject's hypothesis be tenable based on previously obtained information. In Laughlin's procedure, non-redundant information may be obtained by altering one attribute from the initial instance whose relevance is not yet determined (conservative focus) or altering more than one attribute (focus gamble) if either the instance was positive, or else the ambiguous information was correctly resolved on the next choice by altering only one attribute. In the present study Laughlin's *behavioral focusing* (BF) index is computed by applying these rules, setting (BF) = f/C, where f is the number of focus choices, and C is the total number of choices to solution.

In our study subjects were asked after each card choice the reason for that choice. A choice is defined as an *intended focus* if the subject mentions an attribute (or attributes) whose value was changed in the choice from the initial positive instance; it is an *intended scan* if the subject gives no reason or in the reason makes reference only to attributes whose values are unchanged in the choice from the initial positive instance; it is *relational* if the subject states in the reason a relation between the choice and the initial positive instance rather than mentioning shared characteristics; it is *unscored* if the experimenter forgets to ask the reason for the card choice. The intended focusing (IF) index and intended scanning (IS) index are the ratios of the numbers of intended focus and intended scan choices to the total number of choices to solution.

DESIGN OF THE STUDY

The study employed a 3x5 orthogonal factorial design, with three task groups (A,B,C) and five experimenters. Each group received two successive tasks: Group A, Task 3 then Task 1; Group B, Task 2 then Task 1; Group C, Task 1 then Task 3. Subjects were 108 female college undergraduates, with 36 assigned at random to each group. The five experimenters administered tasks to 36, 24, 18, 18, and 12 subjects respectively, divided equally among the groups. Sessions were taped and transcripts were prepared; these were used in conjunction with experimenters' written records to calculate scores. For each subject on each of the two problems, six scores were calculated: (C) number of *choices* to solution; (T) *time* in minutes to solution; (S) *solved* [1] or not solved [0] in 20 minutes; (BF) Laughlin's *behavioral focusing* index; (IF) *intended focusing* index; (IS) *intended scanning* index. Multivariate analyses were performed to test for overall differences on the basis of experimental group, experimenter, and experimenter x group interaction. Group differences were then examined by univariate methods, as summarized below.

FINDINGS

The Hotelling-Lawley MANOVA test was significant ($p = .0001$) for overall group differences, but not for experimenter differences or experimenter x group interactions, as desired. All experimenter groups are therefore pooled in reporting the data in Tables 1 - 7 below. For reasons discussed above, the (BF)-(IF) correlation for each group solving each task was of great interest to us. These were: Group A (-.07,-.11); Group B (.22,.11); Group C (-.21,.25); none were significant ($p > .10$). In the tables, we see that the task variables affected the problem difficulty as expected (Table 2), experience with Task 2 or Task 3 first improved performance on Task 1 (Table 3), and the task variables affected both the behavioral and intended strategy scores (Table 2). In addition, intended strategy use was considerably more influenced by prior experience on a related problem than was behaviorally measured strategy use (Tables 4, 5, 7). Neither intended focusing nor intended scanning was significantly more effective on any task, while a high behavioral focusing index was associated with problem-solving success (Table 6).

Table 1. *Mean Values of Task Difficulty Variables (C), (T). (S), and Strategy Scores (BF), (IF), (IS), for Each Experimental Group Performing Each Task*

Dependent Variable	A (N=36) Task 3 First	A (N=36) Task 1 Second	B (N=36) Task 2 First	B (N=36) Task 1 Second	C (N=36) Task 1 First	C (N=36) Task 3 Second
(C)	14.47	5.28	13.75	6.97	8.67	14.53
(T)	16.32	3.40	13.26	4.86	7.18	13.60
(S)	.36	1.00	.56	.97	.97	.53
(BF)	.29	.64	.34	.57	.56	.32
(IF)	.06	.15	.20	.27	.39	.19
(IS)	.87	.84	.75	.73	.61	.78

Key: (C) number of *choices* to solution; (T) *time* in minutes to solution; (S) *solved* [1] or not solved [0]; (BF) Laughlin's *behavioral focusing* index; (IF) *intended focusing* index; (IS) *intended scanning* index

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Table 2. *Main Effects and Differences Among Groups for the First Problem (A = Task 3, B = Task 2, C = Task 1)*

Dependent Variable	SS	df	MS	F	p	Means A	Means B	Means C	Group Differences[a]
(C)	720.80	2	360.40	8.57	.0004	14.47	13.75	8.67	A,B > C
(T)	155794.89	2	77897.44	18.08	.0001	16.32	13.26	7.18	A,B > C
(S)	7.02	2	3.51	19.06	.0001	.36	.56	.97	A,B < C
(BF)	15469.02	2	7734.51	12.59	.0001	.29	.34	.56	A,B < C
(IF)	19613.63	2	9806.82	23.91	.0001	.06	.20	.39	A< B< C
(IS)	12217.35	2	6108.68	13.07	.0001	.87	.75	.61	A> B> C

[a] Results of Duncan's multiple range test ($\alpha = .05$).

Table 3. *Main Effects and Differences Among Groups for Task 1 (A = Task 1 After Task 3, B = Task 1 After Task 2, C = Task 1 First)*

Dependent Variable	SS	df	MS	F	p	Means A	Means B	Means C	Group Differences[a]
(C)	206.72	2	103.36	4.72	.0111	5.28	6.97	8.67	A < C
(T)	26174.46	2	13087.23	5.89	.0039	3.40	4.86	7.18	A,B < C
(S)	.02	2	.01	.48	.6299	1.00	.97	.97	
(BF)	1467.80	2	733.90	1.12	.3313	.64	.57	.56	
(IF)	9499.57	2	4749.79	6.18	.0030	.15	.27	.39	A < C
(IS)	9546.69	2	4773.35	6.22	.0029	.84	.73	.61	A > C

[a] Results of Duncan's multiple range test ($\alpha = .05$).

Table 4. *Main Effects and Differences Among Groups for Task 3 (A = Task 3 First, C = Task 3 After Task 1)*

Dependent Variable	SS	df	MS	F	p	Means A	Means C	Group Differences
(C)	.06	1	.06	.00	.9730	14.47	14.53	
(T)	13366.13	1	13366.13	2.97	.0900	16.32	13.60	
(S)	.50	1	.50	1.95	.1673	.36	.53	
(BF)	153.15	1	153.15	.40	.5270	.29	.32	
(IF)	3254.01	1	3254.01	16.50	.0001	.06	.19	A < C
(IS)	1431.13	1	1431.13	5.96	.0175	.87	.78	A > C

Table 5. *Main Effects and Differences Among Groups for Tasks 1 and 3 Combined (A = Task 3 then Task 1, C = Task 1 then Task 3)*

Dependent Variable[a]	SS	df	MS	F	p	Means A	Means C	Group Differences
Σ(C)	213.56	1	213.56	2.54	.1163	19.75	23.20	
Σ(T)	2005.56	1	2005.56	.25	.6217	19.72	20.78	
Σ(S)	.35	1	.35	1.32	.2542	1.36	1.50	
Σ(BF)	480.50	1	480.50	.38	.5409	.93	.88	
Σ(IF)	24053.56	1	24053.56	18.53	.0001	.21	.58	A < C
Σ(IS)	18304.22	1	18304.22	14.85	.0003	1.72	1.40	A > C

[a] The variables here are the sums of the scores on the corresponding variables for the two problems separately. Thus Σ(C) and Σ(T) are the total choices and time to solution respectively; Σ(S) is the number of problems solved by the subject. Σ(BF), Σ(IF) and Σ(IS) are twice the average strategy score for the subject on the two problems, weighting each problem equally; these differ from the weighted averages which would be needed to define combined strategy indices as fractions.

Table 6. *Pearson Product-Moment Correlations Between Measures of Performance and Measures of Strategy Usage on Each Problem, by Experimental Group*

(a) First Problem

		(C)	(T)	(S)
Group A (Task 3)	(BF)	-.71[a]	-.79[a]	.63[a]
	(IF)	.07	.01	.31[c]
	(IS)	-.26	-.31[c]	.11
Group B (Task 2)	(BF)	-.74[a]	-.75[a]	.51[b]
	(IF)	-.29	-.02	.04
	(IS)	.24	-.16	.16
Group C (Task 1)	(BF)	-.84[a]	-.79[a]	.24[e]
	(IF)	.17	.14	-.20
	(IS)	-.18	-.15	.20

(b) Second Problem

		(C)	(T)	(S)
Group A (Task 1)	(BF)	-.70[a]	-.58[a]	---[d]
	(IF)	.13	.14	---
	(IS)	-.15	-.17	---
Group B (Task 1)	(BF)	-.81[a]	-.80[a]	.27[e]
	(IF)	-.20	-.06	-.07
	(IS)	.19	.05	.07
Group C (Task 3)	(BF)	-.76[a]	-.82[a]	.61[a]
	(IF)	-.16	-.13	.25
	(IS)	.15	.04	-.15

[a] $p < .001$ [b] $.001 < p < .005$ [c] $.05 < p < .10$ [d] all solved [e] nearly all solved

Table 7. *Pearson Product-Moment Correlations Between Strategy Scores on Successive Tasks for Each Experimental Group*

	BF-BF	IF-IF	IS-IS	IF-IS	IS-IF
Group A (Task 3-Task 1)[a]	-.13	-.17	-.31[b]	.18	.32[b]
Group B (Task 2-Task 1)	.41[c]	.40[c]	.25	-.40[c]	-.24
Group C (Task 1-Task 3)	.22	.54[d]	.46[e]	-.46[e]	-.53[c]

[a] Recall that here there was almost no intended focusing on either task.
[b] $.05 < p < .10$ [c] $.01 < p < .05$ [d] $p < .001$ [e] $.001 < p < .005$

DISCUSSION

It should not be assumed that the tasks we administered, in which subjects are questioned after each card choice, are identical to tasks without such questions. Nonetheless it appears that Laughlin's behavioral focusing index and our intended strategy indices measure very different things. The former with some degree of logical necessity correlates highly with success, while the latter do not; the latter are more influenced by experience on the prior problem. Finally, the (BF)-(IF) correlations are small and insignificant. These findings suggest to us that "intended strategy" scores are the more informative measures, and deserve detailed investigation. It should not be taken for granted that the *intended* use of focusing is associated with success. In addition, the near-disappearance of intended focusing as the tasks come more closely to resemble real-life situations suggests a more fundamental role for scanning in our understanding of every-day concept acquisition strategies.

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THE PURSUIT OF MATHEMATICS TEACHERS' BELIEFS

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A description of our two year project was provided in the following abstract which accompanied the proposal to the National Science Foundation.

> *This project involves the explication of a scheme for describing teachers' conceptual systems, a description of teachers' conceptual systems using the scheme and a description of how those systems relate to instructional decisions in both the planning and interactive phases of teaching. Conceptual systems will focus on three aspects of teachers' belief systems: What they believe to be mathematics; what they believe to be purposes for teaching mathematics; and what "theories," either implicit or explicit, they have for explaining how mathematics is learned and taught. The relationship between the conceptual systems and instructional decisions such as content selection, selection of pedagogical approach, and emphasis given to various types of mathematical outcomes will be investigated. An interdisciplinary approach in which both the literature and scholars from a variety of fields will be used. Teachers will be interviewed and case studies conducted to study the relationship between conceptual systems and decision making.*

A basic premise of this project is that teachers' actions are predicated on their conceptual systems, i.e., their beliefs about mathematics and the teaching of mathematics. Our efforts in the first year of the project focused on issues related to studying such beliefs. This paper is as much about the processes we have struggled with in trying to fulfill our initial aims as it is with "conclusions" based on our first year's work.

THE WAY WE BEGAN

Though for the NSF proposal we had reviewed relevant empirical research, we began our work on the project per se with a search for perspectives that were less empirical. As a result we encountered such works as Schiro's Curriculum for Better Schools, Scheffler's Conditions of Knowledge and Reason and Teaching, Green's Activities of Teaching, Lakatos' Proofs and Refutations, Rokeach's The Open and Closed Mind and various treatises on the nature of mathematics. We then decided to develop what we termed "episodes" to stimulate interview discussions. The episodes were designed to promote discussions on what constitutes mathematics, why mathematics should be taught, and how mathematics is best taught and learned. These episodes were our tools to "get at" teachers' beliefs about the aforementioned areas. Some involved direct questions such as, "How did you teach the topic?" or "What would you most like your students to learn by the time they leave your class?" Others required the teacher to react to a situation in which they must project themselves into some role. The following episode is typical.

> The top student in a high school is interviewed by the local press. During the interview she indicated that she enjoys both mathematics and literature. Indeed they are her two favorite subjects. But she claims that she likes them each for very different reasons. Literature is appealing because it is open, exploratory, never finished. Math on the other hand is clear, concise and such that when you've finished a problem, you are ready to move on to a new one, but you are not troubled continously by what you've done already. What do you think about her answer? If you were to interview her further what new question would you ask?

Other episodes asked the teacher to select stated positions which best reflect their reasons for teaching mathematics, problem solving, or whatever. Another set of episodes involved imagination as indicated below.

> If you could be a famous person when you were teaching, what person would you pick? Why?

Episodes (as opposed to direct questions) were used to enable us to learn, in a fairly oblique and nonthreatening way, about teachers' beliefs. We hoped their use would promote open discussions and minimize their responding with idealized notions. We did experience some tension, however, over whether the episodes would provide us with answers or with entrees to conversations.

THE INTERVIEWS

When the interviews began, we anticipated that they would be based upon one or two sessions with approximately ten teachers, making use primarily of the aforementioned episodes. After completing two one hour sessions with four teachers using predetermined episodes, we began to realize the complexity of the task and decided it would be beneficial to conduct more extensive interviews with the same subjects and to move beyond responses to episodes. As a result we spent two or three interviews with each teacher to probe, clarify and extend responses to the initial episodes presented. Basically the first four sessions were devoted to discussing initial reactions to various episodes and subsequently following up on those responses.

Since our interest is primarily how teachers negotiate meaning and since meaning necessarily entails relationships and not discrete, one shot ideas, we devised a scheme which would permit us to identify what the teacher perceived as important and how those ideas would cluster together to form beliefs. Each teacher was asked to reflect on the transcripts from the first (approximately) four interviews, to identify his/her statements which were deemed important and to write those statements in green ink on a 3x5 card -- subsequently called "green cards." This permitted us to separate out those statements which were significant to the teacher amidst the enormous volume of material we were generating. Each teacher was then asked to group the green cards in clusters, the criteria for clustering being loosely defined with the directive, "Which ones seem to fit to you?" The teacher was then asked to create a category together with a brief description (called a "red card") for each group of green cards clustered together. An example is given below.

Red Card: STUDENTS AND PROBLEM SOLVING
(The student as a problem solver will hopefully foster creativity and analytic thinking)

Green Cards. i The answer is not important but rather the understanding what the problem is asking, analyze the information and write down bits of information that mean something to the problem.

ii Problem solving and applications help students develop strategies to cope with problems

iii I think students should be exposed to thinking of math in a creative sense - looking at something from a different perspective.

iv I believe there are many good reasons for teaching problem solving

Some teachers found the clustering and categorizing of statements to be a fairly straightforward task. At least one teacher, however, evinced considerable frustration during the clustering process and in stating a category and associated descriptor.

The process of forming green and red cards took place over several interview sessions. The final session focused on trying to discover what factors contributed to the beliefs expressed during the previous sessions. We refer to this phase of investigation as our "origin phase." A number of different strategies were used in order to determine the extent to which teacher education programs influence the beliefs of the teachers and the extent to which other individuals or professional experiences were influencing factors. Some of us asked the teachers to cluster the green cards once more making use of origin as the criterion. Others asked the teachers directly what accounted for their beliefs. In addition we occassionally asked teachers to role play their life as elementary school students. The responses to these inquiries provided little surprising information and were only tangentially included in our "Summary Reports."

THE TEACHERS

With respect to beliefs mathematics teachers hold, we have heard it said by at least one esteemed colleague that "math teachers for the most part all believe the same thing about teaching and learning mathematics." We believe we have at least a strong existence proof to the contrary. The four teachers Fred, Janice, Mary Jane, and Marsha all demonstrated quite unique beliefs which we believe would influence their classroom behavior in quite different ways, if one is willing to consider something other than the more superficial aspects of teaching. While it is nearly impossible to capture each teacher's beliefs as expressed in over seven hours of interviews in a brief page or two, we will try to indicate ways in which these teachers are highly idiosyncratic in their beliefs.

Janice

Janice is in her mid-twenties, has one year of teaching experience, is married and the mother of a young child. She is a very open person who enjoyed responding to the episodes -- "They make me think about things." Janice often espoused "current" mathematical education jargon as she proclaimed the importance of problem solving and applications. However, when pressed for specifics, she could not identify specific problems or applications that she used and, indeed, indicated that was an area in which she needed help.

It seemed that Janice's conceptual systems were more grounded in social/psychological considerations than subject matter oriented ones. She identified her following statements as particularly important.

I would like students to feel better about themselves as math people.
There are just too many distractions be it in the classroom or school or social environment.
Some students have difficulty because there is a hard adjustment for them to come out of middle school into high school.
I would like to eliminate all negative interaction from other people.
An ideal teacher is a personable person -- just someone that the students enjoy being around.

Janice indicated that her father was an influencing factor in her choosing to be a mathematics teacher because, "He was relaxing in the math situation."

In general, Janice is a teacher who views herself as a caring person who wants to promote a caring and supportive classroom atmosphere. She does not seem to be driven by overriding academic concerns. When asked to describe four students who had difficulty in learning mathematics she emphasized a lack of motivation and poor interpersonal relationships. Seldom did she mention problems inherent in the content (e.g., some topics are more difficult than others) or lack of prerequisite knowledge.

Fred

Fred is 26 years old, married, and is working on a Master's degree in mathematics education. He sees himself as a quiet person but one who is a "people person." He views himself modestly as a person of slightly above average intelligence. He believes that problem solving is the essence of mathematics, that heuristics should be the central focus of teaching problem solving and that applications are very important. He maintains that his view of mathematics has changed from a "do rules and find answers" perspective to a "maybe there are no definite rules to find a solution -- maybe there is no solution." Despite his professed belief in the importance of problem solving and applications, he was unable to identify specific problems, applications or techniques used to teach problem solving and applications. Fred recognized this "deficiency" and attributed it to the fact that his teaching experience consisted of only student teaching.

Fred's description of the ideal teacher includes approaching lessons in a creative manner, staying calm, reacting in wise ways, knowing content, and being interested in student questions. He seems most concerned with cognitive outcomes such as knowledge of rules and ability to use heuristics in problem solving.

Fred would like to be a Polya ("Polya could field almost anything. He doesn't get stumped.") or an Einstein ("Who wouldn't want to be a genius?"). He is excited about the potentials of a problem solving emphasis but seems somewhat naive about the realities of the classroom. He relies on both authority (e.g., professors) and his own experiences for construction and validation of his beliefs. (Often his discussions seem to be playbacks of content learned in mathematics education courses.)

In general, Fred would prefer to act as a result of thought rather than emotion. When asked a question, he would think about the question, respond, and then stop. The clustering of cards was an easy task for him -- he was quite decisive. Most of his red cards reflected an emphasis on problem solving and a rather cognitive orientation although he did emphasize the importance of motivating students and making mathematics fun to learn. Fred occasionally evinced some discomfort in responding to questions. This uneasiness, he later explained, was based on a certain amount of apprehension over the fact that his answer might appear to be inadequate. It is important to Fred not to "look dumb" in front of a class and tends to shy away from situations where that might be a possibility.

Mary Jane

Mary Jane is a Roman Catholic Sister in her early forties who has been teaching high school mathematics for sixteen years, at a small, private, parochial school located in the heart of an urban/industrial area. She has reached an important decision in her life, namely, not to continue teaching mathematics in a formal classroom environment. She feels that teaching mathematics does not allow her the kind of personal interaction with students which she desires. Hence, she is currently in search of work with either prisoners or unwed mothers who need help in finishing high school equivalency work. She feels this will provide her with the kind of personal contact which she believes is missing in a traditional classroom.

Perhaps the most pervasive characteristic of the interviews was a tone which might be described as either intellectual or rational. We heard much about what she knew, how she understood, and why she believed, but little about how she felt or how she reacts on a more emotional level. When the conversation began to touch her childhood or her insecurities, she evinced some discomfort and quickly steered the conversation away from what appeared to be sensitive areas for her. Yet she was quite talkative and very cooperative; she needed very little prompting or prodding to talk about her ideas.

Mary Jane believes that there is "nothing she can think of that does not have math involved." To her, mathematics must have applications to the real world. She reports that she tries to expose students to how mathematics is used by professionals and business people as often as she can. Although she likes to draw parallels between mathematics and the real world and tells her students about how much there is a "fit" between the two, she could not recall many mathematical applications which she uses in the classroom.

An important belief which Mary Jane expresses has to do with what she calls classroom atmosphere, that is, the "tone" of the class, e.g., formal, informal, structured or loose. She believes that classroom environments are better if they are informal. Students need to feel comfortable in class, she says; this encourages them to ask questions, respond to one another, and have a more positive experience generally. However, other evidence, viz., hearsay and comments made by Mary Jane during the course of the interviews, suggests that students do not perceive her classroom as being informal. Several of her students indicated in an unanticipated talk that she is perceived as a kind of martinet; they do not feel comfortable in her classes. As implied by these students, the problem seems related to a mismatch between her concern with students as emotional beings and a lack of emotionality in the way she relates to people. Perhaps this possible mismatch is a contributing factor in her decision to leave the formal part of classroom teaching.

Marsha

Marsha is 51 years old, a grandmother and widow and has been teaching mathematics in a suburban junior high school for fifteen years. Marsha did not prepare to be a mathematics teacher and has had no preparation in mathematics education save for one course.

It is very difficult to appreciate the significance of Marsha's world view without taking into consideration that she straddles two cultures. Marsha was born in Japan but raised there by Japanese parents who were in fact part of the intelligensia and adopted many Western cultural values. Her father was a journalist and her mother a teacher of English. Though as a child she had the opportunity to see the infantile and morally debasing Abbott and Costello only "on the sly," she was encouraged to read and discuss (with her parents) the works of Shaw, War and Peace, and the lascivious Lady Chatterley's Lover (banned in Japan). She married an American in Japan and emmigrated to the States as a young adult.

Marsha is a wonderful potpourri of contrasts. She talks of teaching as "mothering," and developing a humanitarian attitude while she also strikes some youngsters hard enough on their heads to bring them to tears. She talks of her own inadequacy in taking lecture notes and of preparing youngsters to learn how to do so as an important educational goal. She talks of the paradox of respecting teachers when they in fact may not deserve respect, and of liberating and "weening" students while at the same time requiring that they see things her way.

With regard to both teaching and learning mathematics, Marsha adopts in part a model which has mechanistic components and is rooted in the work ethic as well. She comments,

> When I teach mathematics, I teach them to understand what's going on in every detail.

The point of view is elaborated upon in her description of what kind of activity she expects all students to engage in when she first introduces a topic. She says,

> I ask them to take complete notes and show every detail.

Evidently those who are doing well are expected to learn to pay attention and are not required to take detailed notes in class. But those who have difficulty are expected to improve their performance through "writing everything I told you to write and say in English 'opposite leg' and all that."

With only a trace of irony, she embellishes the theme of mothering by commenting that the prerequisite for good teaching is a "trust for people or some sort of belief in human beings . . . humanitarian type attitude towards children or people." It is interesting that Marsha's justification for mothering and concern with feelings is not only rooted in a humanitarian philosophy but is in part based on personal experience and in fact represents an antidote for an otherwise dull/mechanistic experience.

Marsha spoke about the reward for studying mathematics coming at the end, rather than in anticipation. The enjoyment is postponed. Though one could perhaps live (even if somewhat uncomfortably) with that ethic, she goes one step further when she comments that she appreciated the significance of arithmetic much later in life (so far consistent with her earlier expressed belief) but

> . . .When we got to calculus, it was more interesting, and then after that I thought mathematics was beautiful. But that was much later, and the minute it became beautiful, it became too difficult.

That is, you wait for the appreciation until the end, and as soon as you get a glimpse of it, the phenomenon becomes inaccessible. There was only the promise to begin with, and as soon as it is in view it evaporates.

Thus we see that Marsha is an immensely complex individual who seemingly holds contrasting yet compatable (in her mind, at least) beliefs. On the exterior, rather pedestrian notions abound -- learning mathematics is dull, mechanistic. Yet beneath the surface there exists extensive imagery: learning mathematics is like climbing a mountain, learning poetry is like diving into a pool of wine. Whether her practices in teaching mathematics is mechanistic or imaginative, we do not know. But the beliefs which are foundational to her practice are anything but mechanistic and dull.

BEYOND THE FOUR TEACHERS

There are a number of family resemblances that emerge from even these brief portraits of our four teachers.* Using different language, and sometimes focussing on different nuances of that concern, it appears that all of the subjects are in some sense concerned with the issue of control. Fred, for example, is concerned with control over the subject matter lest he be thought of as mathematically inadequate. Janice would prefer to teach in an atmosphere of love and sweet reason, controlling the occurrences of negative interations among students and between students and herself.

In addition to control, they all have tended to express a view of mathematics as a relatively "cold" discipline, one in need of being made interesting and relevant through a variety of pedagogical techniques. The inability of Mary Jane to overcome a view of the subject as essentially removed from people may have been responsible for her desire to leave the profession. Marsha's concern is less extreme, but she compensates for perception of the discipline as cold by choosing to "mother" her students - thus providing an element of warmth.

It can be argued that these are no great "finds" since it is well documented in the research literature that control is a very basic concern for teachers. Too, we have talked with other teachers who seem to consider mathematics as something detached from people but think of teaching mathematics as something involving human intimacies. So, here we are with our "high-powered machinery," episodes and variations of episodes, essentially seeing what the profession has heard for a long time -- ordinary stuff.

If, however, we are willing to shift the interest within which we see the ordinary, we may reveal new sites to explore, sites which are far from banal. Take the issue of concern with control, for example. Instead of asking whether or not teachers possess such concern, let us focus on the roots of the concern. Does the issue of control in teaching have a source that goes back in time? Is the source of control in teaching the same as it is in other areas? Too, how does a concern over control affect one's methods for teaching mathematics? Is such a concern an extensive limiting factor in presenting mathematics in other than a dualistic, right vs. wrong mode?

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*In this report we have merely highlighted a small number from the wealth of information we have acquired for each teacher. We have selected those observations that raise the potential for next steps in our research. More detail for each subject can be found in our summary reports of June, 1982.

In an attempt to be more perceptive we have also employed the heuristic of looking beyond the content of their beliefs and to focus on the structure or logic of their beliefs. Fred speaks about the importance of problem solving but cannot specify the kinds of problems or applications which promote problem solving behavior. Both Janice and Mary Jane think that applications are an important pedagogical concern, yet each of them is hard-pressed to come up with examples of application on their own.

What we have is a very interesting pattern of teachers expounding strong beliefs in the importance of X and yet not being able to give examples of X as a phenomenon. They have "bought" something at an abstract level and yet are incapable of providing examples of it. Note that it is not the case that they merely have difficulty justifying a belief - something that may be difficult for us to do with our most cherished beliefs - but that they cannot give specifics of the phenomenon they think important.

POSSIBLE EXPLORATIONS

Let us now shift gears from one of exposing finds to that of pinpointing possible research questions that might occupy our thinking over the next few years. In many respects our ability to find wonder in the ordinary (as in the discussion of control earlier) is a significant starting point for research. To find wonder in the ordinary, however, depends very much upon an ability to frame questions. It seems revealing to us that research has become defined more in terms of results, i.e., "finds" than in terms of a new set of questions that might be proposed. For hundreds of years people did research in mathematics in a vain effort to prove the parallel postulate. But significant "findings" did not occur until someone changed the question from "How to?" to "Is it possible?"*

There are a number of "events" in the project which constitute puzzlements for us. Some of these suggest possible project activities as we begin our second year.

(1) Espoused Beliefs versus Beliefs in Practice

As described in the introduction, we had intended as our next phase of the project to use the belief system scheme created in the first phase to determine how beliefs impinge on decisions in the classroom. It has occurred to us recently, however, that it may be worthwhile to look at the relationship between beliefs and practice in a somewhat different way as well. Instead of assuming a causal connection between the two, we might choose to look at those occasions when espoused beliefs do not mesh with "beliefs in action". An entree to puzzlements regarding such apparent incongruities could include an examination of the nature of teachers' decisions in order to better understand their beliefs.

(2) The Hanging Together of Beliefs

We can view belief systems for any teacher from two different perspectives. We could expose both the structure and the logic of the beliefs they hold with regard to any one of the four dimensions we identified at the beginning. It was our original intention to do so with regard to the content of their beliefs but though we have told parts of that story, more remains for us to analyze. Perhaps a return to the transcripts might reveal that there is more to be found than we have discovered in even the longer reports written on each teacher. Perhaps we should concentrate not only on how beliefs within our four specified areas hang together but also on how those beliefs relate to beliefs more generally held, e.g., those about religion.

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*The issue of the role of question asking and of finding wonder in the ordinary more generally is discussed at greater length in the document by S. Brown of September., 1982 entitled, "The Empirical Nature of Our Study."

(3) The Interviews Revisited

We have seen numerous strengths and weaknesses of our interview process and the utilization of episodes. Some episodes seem to stimulate "richer" discussions than others. We have, in fact, a total of over 40 episodes which focus on the four areas of interest and which use a variety of tools for promoting discussions. We could envision a productive review of both how and why the episodes work and of our methodology more generally speaking. In general, we feel the episodes offer an innovative tool for qualitative research in mathematics education and could envision their refinement and use for both researchers and practitioners.

(4) Research and Researcher

We have come to appreciate how much the questions we ask and what we perceive are functions of our personality and our mind set both as individuals and as a team. Frequently investigators attempt to discount these variables and speak of them as biases - to be avoided. We have come to appreciate, however, that the intuitions and prejudices of a researcher are not merely biases and not merely something to be documented as a warning to the research community, but represent a potential source of strength and is something to be studied. We are in search of a language to do so.

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SEX DIFFERENCES, CLASSROOM INTERACTION AND MATHEMATICS EDUCATION

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Study of the causes and consequences of sex-related differences in mathematics is an active area of research in mathematics education. One approach used in that research examines teachers and classrooms for possible causes of sex differences. Some differences between teacher-girl interactions and teacher-boy interactions have been found. Research on classroom interaction may soon identify teacher behaviors which are related to differential achievement gains by females and males. More research is needed to confirm the identity of such teacher behaviors and then to intervene with teachers on the relevant behaviors to see if equity in mathematics achievement outcomes for females and males is attained.

This paper presents highlights of research on sex-related differences in mathematics which has focused on mathematics classroom processes. During the last 10 years interest in educational equity for females and males has increased, and during these years mathematics has been identified as a subject where inequities exist.

BRIEF BACKGROUND

In some high schools there are sex differences in mathematics course enrollment. In some high schools there are sex differences in mathematics achievement even when differential enrollment is controlled. Sex differences in confidence in one's ability to learn mathematics and perceptions of the usefulness of mathematics occur both when achievement differences exist and when they do not (Fennema & Sherman, 1977). The achievement differences are more pronounced as students proceed through high school and for higher cognitive levels of mathematics such as application or problem solving tasks (Fennema & Carpenter, 1981). At the middle school level all students take the same number of mathematics courses, and in many case take the same courses. There are few sex differences in mathematics achievement but some sex differences are apparent for affective variables (Fennema & Sherman, 1978). At least through grade 8, girls and boys take the same mathematics classes in the same classrooms with the same teachers. Nevertheless, some sex differences exist with boys achieving at a higher level than girls, particularly on high level cognitive tasks (Fennema & Carpenter, 1981). In the late 1970's Fennema (1977) hypothesized that the major variable which causes sex-related differences in the learning of mathematics is the differential number of mathematics courses taken by females and males. More recent research has shown that while enrollment is very important, there are notable sex differences (e.g., application tasks) that occur even when enrollment is controlled (Armstrong, 1981; Fennema & Carpenter, 1981).

Care must be taken not to over generalize these results. All female students do not achieve in mathematics at lower levels than all male students. All female students do not have less positive attitudes toward mathematics than all male students. Even so, a problem does exist and it is important for researchers in mathematics to study the problem.

What causes these sex-related differences in achievement and attitudes? There are many hypotheses concerning this question, ranging from overt sex discrimination to sex-role socialization to innate ability. In actuality there are probably a set of causes that act together to produce sex differences in mathematics. Fennema and Sherman (1978) conclude from their research that an important place to look for these causes is in classrooms and schools. Since students learn most mathematics in school classrooms from their teachers, it is likely that classroom processes are related to the development of sex differences in mathematics. Though the society at large and the family are undoubtedly important in molding students' attitudes and expectations, it is teachers that spend time daily with girls and boys working on mathematics. Research on teaching has confirmed the importance of teachers in the amount students learn. Both correlational and experimental studies indicate that teacher behaviors are related to gains in student achievement in mathematics (e.g., Good & Grouws, 1979). Teachers make a difference in how much mathematics boys and girls learn.

What role do teachers play in the differential achievement in mathematics by females and males? This is a broad question and the rest of this paper will examine some studies that are relevant to the question.

Three studies and some of their results will be presented.

SEX DIFFERENCES IN TEACHER-STUDENT INTERACTION

Becker (1981) was first to specifically examine teachers' classroom behaviors and sex-related differences in mathematics. Through observational techniques, she studied the ways high school geometry teachers did or did not differ in their interactions with female and male students. Ten high school geometry classes were observed for two weeks, one week early in the school year and one week at the end of the first grading period. For half of each class period, the researcher recorded teacher-student interactions for all students. During the other half period the researcher collected ethnographic data about the class as a whole.

Three of six tests on the teacher-student interaction variables showed significant sex-related differences with males having higher frequencies for the following categories: open questions (student volunteered), higher cognitive level questions by teachers, and callouts to teacher questions. No sex differences were found for direct questions (student did not volunteer), lower cognitive level questions by teachers, and student initiated contacts. The qualitative analysis of ethnographic data indicated more contacts with teachers for males than for females. Teachers were only half as likely to initiate an individual academic contact with a female student as with a male student. Teachers were more likely to sustain interactions with males than with females. Females received less encouragement than males for academic work.

There is an indication in Becker's study of differential behavior by mathematics teachers toward female and male students. Similar results have been obtained in a large body of research in non-mathematics classrooms (e.g., Brophy & Good, 1974; Safilios-Rothschild, 1979). Teachers tend to have more interactions with males than with females. This knowledge does not necessarily imply that because teacher-male interactions differ from teacher-female interactions the mathematics achievement and attitudes of females and males differ. In fact, significant sex differences for three of six variables observed in ten geometry classrooms is not sufficient evidence to generalize the finding to all mathematics classrooms.

LONGITUDINAL STUDY

The second study to be discussed compared participation in mathematics classroom processes over a three year period by girls and boys with specific characteristics. It was conducted under the direction of Elizabeth Fennema at the University of Wisconsin-Madison. The same students ($n = 43$) were observed in their mathematics classes during the sixth, seventh, and eighth grade years. Approximately equal numbers of girls and boys were selected to participate in the study who scored at or above the mean on mathematics achievement and in the top or bottom quartile on the Fennema-Sherman Confidence in Learning Mathematics Scale (Fennema & Sherman, 1976). Teacher behavior and target student behavior were coded daily during a 15-20 day period by two observers in each observed mathematics classroom. One observer recorded characteristics of interactions between the teacher and individual target students using an adaptation of the Brophy-Good Dyadic Observation System (Reyes & Fennema, 1981). The other observer recorded target student engagement in mathematics and certain characteristics of the type of learning activity in which students were involved (Reyes & Fennema, 1981). Neither the teachers nor the students knew which students were being observed.

Two components of the data analyses for this longitudinal study have been completed (Reyes, 1982). The first component consisted of a repeated measures analysis of variance (ANOVA) for each teacher-student interaction and engaged time variable using grade as the repeated measure and sex and confidence level of student as factors. These ANOVAs were used to test for differences between girls, boys, high confidence and low confidence students across years. There were significant F ratios for sex of student for several of the teacher-student interaction variables. Significant sex differences were found for many of the public interaction variables, few of the private interaction variables, and none of the engaged time variables. In each case where a significant F ratio for sex was obtained, the boys were involved in more of that type of interaction than the girls. No significant difference between high confidence and low confidence students was found for any of the teacher-student interaction variables. Among the engaged time variables there was a significant difference between high and low confidence students in spatial engagement. Low confidence students had a higher percent of time when they were attending to a figure or graph while working on mathematics than high confidence students. The second component of the data analyses to be discussed here consisted of process-product correlations between process variables and gains in students' mathematics achievement and between process variables and gains in students' confidence in mathematics. For both of these sets of correlations the magnitude of the correlations was quite large (many r's as high as .7-.8). The magnitude and the sign of the correlations were consistent within each of the four sample groups but were very different from one sample group to another.

This longitudinal study extends knowledge about the role of teachers' classroom behaviors in the development of sex differences in mathematics. Confirmation of the greater number of teacher-boy as compared to teacher-girl interactions was found. Also found was a more detailed picture of the specific types of interactions where sex differences appear. In terms of understanding the implications for differential achievement and attitudes in mathematics, this study does not present clear evidence. Process-product correlations were computed but do not yield a clear picture.

AUTONOMOUS LEARNING BEHAVIOR

The third study to be presented was a process-product study conducted by Elizabeth Fennema and Penelope Peterson of the University of Wisconsin-Madison during the 1981-82 school year. The data analyses have not been completed so no results are available. The study is included here because of the interesting set of product variables which the researchers call autonomous learning behavior. Autonomous learning behavior has dimensions in both the cognitive and affective domains and consists of performance on high level cognitive tasks, confidence in one's ability and willingness to do high level cognitive tasks. The purpose of the study was to identify teacher behaviors that are associated with growth in autonomous learning behavior for girls and for boys.

Approximately 30 fourth-grade classrooms participated in the study. Students in these classrooms were pretested on autonomous learning behavior. Teacher-student interactions and student learning activities for the same classes were then observed for four weeks. After the observations were completed students were posttested on autonomous learning behavior. Correlations between classroom process variables and gains in autonomous learning behavior variables will be computed separately for girls and boys.

This study is designed to give more of an indication than the other two studies of the role teacher behaviors play in the development of sex differences in achievement attitudes in mathematics. The use of autonomous learning behavior as the outcome is a strength of the study. Success in high level cognitive tasks in mathematics is very important if students are to apply mathematics outside of class and advance through the mathematics curriculum.

CONCLUSION

Research to identify if and how teachers' behavior is related to student sex differences in mathematics has been reviewed. One interesting possibility for further research would engage teachers in a collaborative study to identify important teacher behaviors. Another study which follows from the literature reviewed would be an experiment in which relevant teacher behaviors are changed and the impact of these changes on student achievement and attitudes in mathematics is noted.

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RESEARCH AND PROGRAM DEVELOPMENT FOR SEX EQUITY IN MATHEMATICS: THE EQUALS PROGRAM

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The EQUALS program at the Lawrence Hall of Science assists educators in acquiring methods and materials to attract and retain students in mathematics classes by promoting awareness of the relationship of mathematics background to educational and career opportunities and by increasing teachers' use of manipulative materials and problem solving strategies. It provides a good example of disseminating and field testing research in mathematics education. The willingness of researchers and program developers to share perspectives and to work together has enabled each group to be more effective.

Program development in sex equity in mathematics education provides a good example of disseminating and field testing of research findings. From the beginning, developers of programs to increase young women's participation in mathematics have kept in close touch with the researchers whose work might illuminate their objectives and methods. In turn, the researchers have benefited from access to a forum in which to test their ideas in a "real world" context. This paper reviews the experience of the EQUALS program in working with researchers and developing programs to increase females' participation and achievement in mathematics.

A BRIEF HISTORY

In 1973, sociologist Lucy Sells coined the phrase, "critical filter," to describe the importance of adequate training in mathematics for educational and occupational opportunities (Sells, 1975). This work prompted initiation in 1974 of a "Math for Girls" class at the Lawrence Hall of Science (LHS) (Downie, Stenmark & Slesnick, 1981). In 1975, concerned Bay Area educators gathered to discuss ways that they could encourage young women to stay enrolled in mathematics courses and started the Math/Science Network. "Expanding Your Horizons in Science and Mathematics" conferences for young women were started in 1976 (Koltnow, 1979).

Realizing that direct services to young women could not possibly reach enough students and recognizing the importance of teachers in shaping and changing students' attitudes, LHS started the EQUALS program in 1976, in which teachers and others learn strategies for attracting and retaining girls in the mathematics curriculum (Kreinberg, 1981). Requests for EQUALS materials led to the creation of a handbook for conducting an EQUALS program (Kaseberg, Kreinberg & Downie, 1980) and to a curriculum development project to test new materials (Fraser, 1982). Analysis of the success of EQUALS teachers showed that their ability to change their curricula was related to the extent of administrative support they received, so a program was created to provide information to educational decision-makers. In addition, Family Math was initiated in 1981 to teach low income parents how to work with their children on activities that will complement the efforts of their teachers.

Three themes pervade all of the programs mentioned above: awareness, competence, and encouragement. The programs stress awareness of the importance of mathematics for all students and of the consequences of avoidance of mathematics in educational and occupational achievement. The competence component follows naturally, as motivation must be accompanied by substance if it is to produce change. Encouragement is provided both for teachers, some of whom have less than strong mathematics backgrounds themselves, and for students, some of whom find mathematics difficult. When asking teachers to try new approaches and new activities, encouragement is as necessary as awareness and competence.

Throughout the development of the EQUALS program, research and evaluation have been recognized as important concerns. This paper will describe the ways in which research has been used to define program priorities, increase awareness of local problems, and refine program activities.

DEFINING PROGRAM PRIORITIES

When Sells identified the low participation of women in high school elective mathematics courses as a barrier to their enrollment in the majority of undergraduate majors at the University of California, little research was available to explain why so few had taken advanced mathematics courses. Maccoby and Jacklin's review of psychological sex differences concluded that it was "fairly well established...that boys excel in mathematical ability," (1974, p. 352). Fennema and Sherman (1977) put this statement into context by pointing out that, among students whose mathematics backgrounds were similar, sex-related differences were found inconsistently, were small, and were only found in the presence of sex differences in affective factors.

Encouraged by this observation, the EQUALS program began by identifying enrollment in mathematics courses as a high priority. Clearly students will learn more mathematics by enrolling in more courses; it does not necessarily follow that equal enrollment alone will guarantee equal achievement. Subsequent research has shown that, even when the number and kind of mathematics courses taken are controlled, there are still sex-related differences in performance favoring males (Armstrong, 1981). Not abandoning the goal of increased enrollment, which is still obviously involved in increased achievement, EQUALS looked further into the literature on sex-related differences in mathematics. Finding that a number of investigators had located girls' relative lack of success on problem solving items (e.g., California Assessment Program, 1978), EQUALS increased its emphasis on problem solving and developed new curricular materials with this emphasis (Fraser, 1982). Investigators still find a relationship between achievement and the perception of the usefulness of mathematics (Armstrong, 1981), so EQUALS has continued to emphasize this goal.

AWARENESS OF LOCAL PROBLEMS

Investigation of enrollment in elective mathematics courses, prompted by the research cited above, has yielded mixed results. The National Assessment of Educational Progress (NAEP) found, with a national sample, that sex-related differences in enrollment occur only at the highest level, calculus (Armstrong, 1981). Surveys in California, however, have shown that the sex-related difference appears at the intermediate level -- Algebra II (CBEDS, 1981). Since sex-related achievement differences are found in some schools and not in others, even in the same geographic area and with the same instruments (Fennema & Sherman, 1977; Pulos, Stage & Karplus, 1982), it seems prudent to investigate enrollments on a school-by-school basis as well.

Since 1976, EQUALS has had teachers collect enrollment data for the mathematics courses in their schools by sex. Then it is possible to ascertain where, if anywhere, there is an enrollment imbalance at each school. This procedure can uncover an easily remedied problem, such as the finding that more girls were advised into the two-year Algebra I sequence while more boys were advised into the one-year course. The EQUALS teacher who discovered this at her school was able to serve as gatekeeper and make course selections based on need rather than gender. In other cases, the enrollment data allow an EQUALS participant to document his or her success.

Another piece of research that EQUALS teachers have replicated annually is the "Typical Wednesday When I'm 30" story. Content analyses of the fourth through twelfth grade students' stories indicate that students ignore the reality that most adult males take some responsibility for the care of their homes and families and most adult women work outside of their homes for needed income. Further, girls do not seem to be aware of the fact that a number of occupational choices are open to them. It is quite a revelation to teachers when they find that the job most frequently mentioned by their female students, and often by some of their best students, is secretary.

By collecting their own data, teachers become experts on the situation in their own schools. They can use the data to motivate others to help them and to devise strategies that will make students examine their career plans, particularly the role of mathematics preparation. In addition, teachers gain some appreciation for and interest in research done by others and take greater interest in some of the professional journals. This involvement of teachers in research expands the base of participation and dissemination for research done by others.

PROGRAM REFINEMENT

Much of the research that is done on the EQUALS program is, of necessity, for evaluation purposes. Enrollment data provide evidence of the increased participation in high school mathematics courses, but the program serves teachers K-12, so it is necessary to consider other types of data for evidence of success with younger students. From the literature on precursors of mathematics enrollment, recognition of the usefulness of mathematics in one's future has been repeatedly identified as a critical factor (e.g., Armstrong, 1981). EQUALS has assessed students' recognition of the usefulness of mathematics in two ways -- through a Mathematics Attitude Scale (MAS) and through a Career Interest Survey (CAS).

The MAS is modeled on three scales of the Fennema-Sherman attitude scales (Fennema & Sherman, 1975) -- effectance motivation, usefulness, and confidence. It has consistently shown significant gains for the students of EQUALS teachers at most ages. When comparison groups have been available, the picture becomes clearer, because non-EQUALS students' attitudes go down somewhat over the course of the academic year. This is consistent with others' findings that positive attitudes towards mathematics decline somewhat over time, and more so for females than for males, so that approximately equal attitudes found in elementary grades for girls and boys become disparate during high school (e.g., Brush, 1980).

The CAS consists of a list of job titles and the instruction to check as many of the jobs as you would like to try for a day. In the first year of EQUALS, the comparison of pretest and posttest results showed no change over the course of the year. Staff members realized that it is easier for teachers to cultivate positive attitudes towards mathematics, which most of them possess, than it is to transmit information about a variety of occupations, which most of them lack. As a result, the program staff increased the time given to career information and provided more information on availability of resource materials and role models who might visit the classroom to share their experiences. The results on the CAS for the following year showed that students did increase the number of careers that they would consider.

SUMMARY

The examples above have shown how EQUALS has benefited from its association with researchers in shaping and documenting the program. There is a reciprocal relationship, with equal benefit accrued by researchers who are associated with the program.

ACKNOWLEDGEMENTS

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Can Computerized Instruction Effect Elementary Mental Abilities?

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The purpose of this paper is to examine, briefly, the broad question of relationships among aptitudes and treatments in an emerging educational environment characterized by ready access to and daily use of computers and computer-based technologies. Relationships among aptitudes and educational treatments have been foci of much educational research; in general these studies have examined the roles of aptitudes as predictors of success in achieving various educational tasks or goals given certain treatments or conditions. Following this line of research, and recognizing the need to prepare students to function in a computer-based society, researchers have begun to address a constellation of questions concerning a) what aptitudes predict learning of effective computer use, and b) what educational treatments are effective in teaching students to use computers.

While these questions provide important issues for extensive study, they do not encompass the scope of essential questions raised by the widespread use of computers, for they fail to take into account the unique characteristics of the computer and its potential impact especially in areas related to mathematics education.

The study of these characteristics, or their structural relationships to various aptitudes, and of the potential effects of long term experience with them, suggest further questions. These questions, to be explored below, are derived by viewing aptitudes as products (in part) of long term experience and by asking, in essence, what contribution computers will make to this experience, and thus to resultant aptitudes. The aptitudes of particular interest here are those which have theoretical or empirical relationships to the learning of mathematics.

The Computer Based Learning Environment. The pervasiveness of computers effects virtually every aspect of modern society; indeed many authors argue that the impact of computers on society (and thus on education) will be comparable in magnitude to that of the printing press (e.g., Evans 1979). Educators differ in their assessments of the most salient effect that computers will have on learning. While some (e.g. Papert 1980, Shumway 1982) argue that the teaching of various programming languages (e.g. LOGO, BASIC, PASCAL) and techniques, especially structured programming (Bork 1981), will enhance students learning of logic and their understanding of and approach to mathematics. Others argue that in the near future computer users will have available sophisticated authoring systems and software utilities which can be addressed through natural language; for these writers the educational promise of computers lies in the educational software that will become available.

Both positions rely greatly upon the interactive capability of computers, the ability to respond to user messages in differentiated ways. Frederick Goodman (1981) argues that interactivity affords students an entirely new learning environment, one in which they can send messages which produce effects.

The message (input) and response (output) modes available today include both digital (keyboard, light pen, touch-sensitive screen) and analog (paddles, joysticks, voice recognition) input devices and various complex output devices (text screen, complex graphics, music, voice synthesis). In the out-of-school experience of learners these devices are becoming commonplace and provide a part of the environment in which a large part of school-age children's learning takes place. Experience with computer-generated graphics of great complexity, correlated sound and motion, pattern recognition, maze tracing, sequencing of steps in the solution of problems, and numerous other phenomena which are familiar to the literature of mathematics learning are embedded in the computer-related experiences of the learner.

The Computer Based Instructional Environment. In an 1981 paper, Mitzel argues that educators have only begun to use computers, that the only extensive use made of computers is in the pacing of instruction, and that computer-basededucation must take into consideration the findings of both cognitive psychology and the psychology of individual difference. Damarin (1981) has argued that the development of computer-based mathematics curricula must be based upon analyses of the learner, the content goal, and computer capability in the search for a "best fit" among these three. Both of these authors argue that computer experiences can cause fundamental changes in learners, and that computer-based experiences will have differential effects upon students with different constellations of abilities.

We have recently begun to compare what is fundamental in the content being taught with computer capabilities appropriate for use in instruction in that content and in certain mental operations that seem to be required in some classical tests of mental ability. Common elements appear in these comparisons suggesting that repeated experience with computer-mediated instruction could change students' scores on aptitudes test batteries such as the SAT and the Kit of Factor-References Cognitive Tests (Ekstrom et al 1976), and change the relationships between scores on these tests and measures of actual mathematical performance. Contrary to the commonly held view that aptitude "true scores" are essentially stable characteristics of the indivudual, there is some evidence in the literature that precisely defined aptitudes can be changed through curricular interventions.

Aptitudes and Change. The impact of specific curricula on precisely defined aptitudes has been studied in two dissimilar contexts, coaching and compensatory education. Coaching supposedly makes college applicants more sophisticated takers of college aptitude tests like the SAT. These tests contain a variety of dissimilar item measuring different specific skills. Some item types show larger coaching effects than others do (Pike and Evans 1973), and these differences are closely observed for their economic importance, both to coaches seeking to maximize quick gains and to test designers seeking to limit them.

Compensatory education, of course, has far broader and deeper goals than mere test coaching. In the United States the teachers and their pupils commonly speak the same language and remediation is often aimed at improving pupil scores on the standard verbally oriented school aptitude and achievement tests: on the Stanford Binet, for example, on readiness tests for reading and numerical operations, on standardized achievement tests or even on teachers' evaluations. Perceptual motor tests, like the Frostig or the ITPA may be added for children who are just learning to read (Whimbey and Shaw 1975). The scores on most of these tests, like the SAT, are summed over a variety of item types reflecting dissimilar specific aptitudes. American educators have evinced little interest in remediations' effects on these specific abilities. There is no economic incentive to examine such effects, as there is in coaching; the theory behind the remediation is often broad-gauge; and in any event the theory may concern deficiencies in the child's linguistic environment rather than deficiencies in the child's own cognitive processing.

In Israel, by contrast, the languages of the teacher and of the (newly immigrated) pupil are often different and non-verbal content free tasks may play a large role in the remedial curriculum. Feuerstein (1980) focuses diagnostically on the specific characteristics of the "culturally deprived" child's cognition and provides specific remedies. The effects of such remediation are evaluated with a battery of corresponding specific aptitude measures - Thurstone's (1965) Primary Mental Abilities.

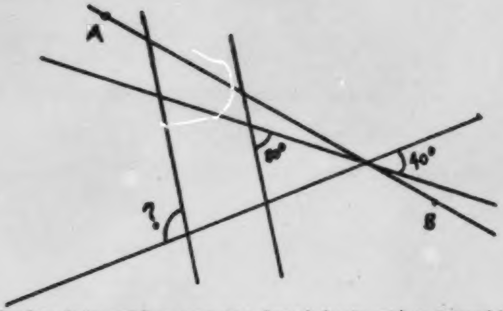
The instructional use of computers provides the opportunity to enrich the experience of the learner with respect to specific aspects of cognition. Moreover, aptitude related "treatments" can be inherent in computer-based activities.

Some Theoretical Applications. The Kit of Factor-Referenced Cognitive

Tests (Ekstrom et al 1976) consists of 72 tests loading as markers on 23 factors, and is an updated version of earlier editions (French 1954, French et al 1963). Several of the factors and tests in the 1963 version have been used frequently as predictors in recent studies of mathematical performance; these include Visualization, General Reasoning, Flexibility of Closure, and Spatial Orientation (Carry 1968, Webb and Carry 1975, Eastman and Carry 1975, Behr and Eastman 1975, Gray and McDaniel 1977, Becker and Young 1978, Young and Becker 1979, Adams and McLeod 1979). While the results of these studies have been mixed (possibly due to the diversity of mathematical tasks used as dependent variables) the overall trend of the findings supports a relationship between these factors and the learning of mathematics. Other tests and factors in the Kit contain items with clear relationships to the mathematics of the schools (e.g. Number Facility, Figural Fluency, Spatial Scanning). Items from tests on these factors are familiar to teachers because they are similar or identical to items used in mathematics instruction.

Many of the markers for these (and other) factors share characteristics with curricular and recreational uses of computers. The depth of the resemblances suggest that extensive use of computers might effect scores on the factor tests, the factor structure of the battery, and the relationship between scores on the tests and mathematical performance. Consider, for example, the factor Flexibility of Closure.

Flexibility of Closure is measured by three tests in which the subject is asked to find certain polygons within complex figures (hidden figures) to find tree diagrams (e.g.) within complex figures (hidden patterns) and to reproduce polygonal paths on a 5 by 5 grid (copying test). The class of geometric problems exemplified by the figure below is solved by finding within the figure certain generalized figures or patterns (parallel lines, vertical angles, triangles) and applying (known) theorems to these patterns.



In designing interactive computer-based instruction toward the objective that students learn to solve such problems, a series of subgoals and related subroutines would be identified; one might provide the student the opportunity to remove irrelevant lines such as AB, to color pairs of parallel lines, to color triangles, to slide lines to compare alternate interior angles, and so on. If the original figure is stored in memory it can always be recalled; therefore the student could examine parts of the figure, draw partial conclusions and return to the full figure (where these conclusions could be recorded for further use). After the student understands and has practiced the task scoring systems rewarding speed and the absence of unnecessary moves could be used to motivate students to try to focus on the essential features without using the computer "helps." Moreover, the use of figures which are randomly generated within (progressively liberal) constraints would guarantee that students would work with a large variety of figures. The general approach described here could be applied to numerous other geometric concepts and problems especially if one adds rotation, reflection, and translation to the repertoire of computer moves available to the student.

While the explicit goals of the instructional sequence are clearly geometric, implicit in the use of the activities is extensive practice in identification and disembedding of simple figures within complex ones. It could be expected that this experience would result in increased scores on the Hidden Figures and Hidden Patterns tests, but not necessarily on the Copying test (although other computer-based activities might effect copying scores).

Other factor markers are also potentially susceptible to effects of computer experience. The Induction factor is marked by concept formation tests and could therefore be effected by extensive use of courseware which is based on the concept learning research (at least three developers have such courseware available). Simulations designed for instruction require the user to select and organize relevant information, and to consider several variables in combination, if used extensively and over a wide range of subject areas might effect the General Reasoning and Integrative Processes factors.

The Spatial Scanning factor would seem to be related to performance (and practice) on the popular maze games (e.g. Pac Man) as well as to some instructional programs, while Flexibility of Use is required in computerized adventure games. Hypothesized relationships of all 23 factors to use of computer software as well as to learning to program are available from the authors as a working paper.

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USING THE MICROCOMPUTER TO TEACH PROBLEM SOLVING SKILLS

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This paper describes a project that uses the microcomputer to provide a means for directly relating formal symbolic representations to children's informal modeling processes. A program has been developed that allows children to use a microcomputer rather than physical objects to solve word problems. They produce pictorial sets of objects one at a time and can make a single set, or make two sets, or remove elements from a set they have constructed. The connection between the informal modeling processes and the formal mathematical symbolic representations is made by teaching the children that they do not have to construct sets one element at a time; they can construct them by writing number sentences. Results of a pilot study carried out with four first-grade children indicate that the program is effective in teaching representational and problem solving skills. Before instruction, the four children consistently wrote incorrect sentences for more difficult problems and generally did not use their number sentences for their solution. Following instruction, three of the four consistently used number sentences to solve a wide variety of addition and subtraction problems.

A microcomputer research project currently being carried out at the Wisconsin Center for Education Research is investigating the transition phase in children's learning of symbolic representational skills in mathematics as they progress from use of informal strategies to writing of formal mathematical sentences to represent verbal problems. A specially produced microcomputer program is being studied in a teaching experiment to see if it is effective in helping to establish a direct link between the formal and informal aspects of children's mathematical behavior.

BACKGROUND

In the last few years a substantial body of research has focused on the learning of addition and subtraction concepts in general and on the solution of addition and subtraction word problems in particular (Carpenter, Blume, Hiebert, Anick, & Pimm, 1982; Carpenter, Moser, & Romberg, 1982; Riley, Greeno, & Heller, 1982). Currently there is good agreement regarding the basic characterization of addition and subtraction word problems, and there is a reasonably consistent picture of the difficulty level of different types of problems and the informal problem-solving strategies children invent independently of instruction. However, relatively little is known regarding the transition from these informal strategies to the formal addition and subtraction skills taught in school (Riley et al., 1982).

A key aspect of the transition from solving problems using informal procedures based upon simple representational skills to a formal mathematics approach is writing mathematical symbols to represent the problem and its components. At the time children are first introduced to writing mathematical sentences to help solve word problems, their informal strategies and procedures make more sense to them. As a consequence, they see no connection between the two activities. The operations represented by the number sentences are often inconsistent with the modeling and counting strategies used to solve the problem. Writing a number sentence is something that young children do for the teacher, something they often perceive as unrelated to the solution of the problem.

In a study investigating the effects of initial instruction on the processes children used to solve basic addition and subtraction verbal problems, Carpenter, Moser, and Hiebert (1981) considered the role of writing number sentences in the solution process. Prior to instruction 43 first-grade children were individually tested on a variety of addition and subtraction word problems. After a two-month introductory unit on addition and subtraction, the children were retested. On the posttest most children could write number sentences to represent addition and subtraction problems. However, very few recognized that the arithmetic sentence was a mechanism that they might use to help them solve the problem. Once they had written a sentence, most children appeared to ignore it and used the semantic structure to decide on a solution strategy. The fact that sentence writing did not influence children's solution processes suggests a lack of coordination by them between the two processes.

DESCRIPTION OF THE COMPUTER PROGRAM

The computer program used in the study was written in Apple Pascal using the Pascal ANIMATION Package from Apple Special Delivery software to help create a large character set and handle some of the display tasks. It requires a 48K Apple II+ with an extra 16K RAM card in slot 0 for operation, and a color monitor. A complete description of, as well as a listing of the program may be found in Moser and Carpenter (1982).

The major feature of the program is the ability to enter onto a video display pictorial and symbolic configurations. The display is arranged in three adjacent sectors which can be thought of as corresponding to the elements of the number sentence $a + b = c$ or $a - b = c$. Pictorial configurations consist of small squares arranged in a pattern resembling the TILE configurations used in Japanese elementary education (Hatano, 1982). See Figure 1 for an example of the configuration. Entry of the pictorial configuration occurs in two ways -- a one-by-one incremental entry effected by depression of the $\boxed{\rightarrow}$ key or by a depression of appropriate numeral key(s). Space limitations confined the number of squares in a sector to a maximum of 30. Squares appear only in sectors a and b. Removal of squares from a sector is carried out by depression of the $\boxed{\leftarrow}$ key. The "take-away" action can be simulated by entry of squares into sector a and then movement of a subset of those squares to sector b, such movement being effected by initial depression of the - (minus) key followed by depression of the $\boxed{\rightarrow}$ key as many times as required or by depression of the numeral key.

Standard mathematical symbols for numbers, operations, equality, and an unknown quantity ($\square$) can be produced by depression of the appropriate keys as needed. (? produces the $\square$) These symbols appear in the lower one-third of the video display as indicated in Figure 1. The

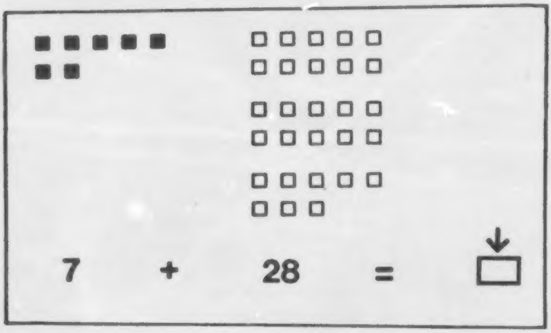

Figure 1. Video screen display after entry of 7 + 28 = $\square$

program calls for the production of only numerals, or for a complete number sentence that is essentially correct in form. Incomplete sentences such as $5 + 7 \square$ or $5 \quad 7 = \square$ would not appear. Sentences that are impossible to solve within the domain of whole numbers such as $13 - 15 = \square$ or $9 + 3 = \square$ would not be accepted by the computer. Sentences, either open or closed, of the form $a \pm b = c$ could be entered except for noncanonical subtraction sentences of the form $\square - b = c$ or $a - \square = c$.

THE TEACHING EXPERIMENT

Four first-grade children, two male and two female, of middle range of ability participated in a five-week teaching experiment in April and May, 1982. Individual interviews were conducted with each subject in which the child was asked to solve the following 10 verbal problems: Join, Separate (2), Compare (add), Compare -- find the difference (2), Join -- missing addend (3), and Separate -- missing minuend. For one each of the Separate, Join -- missing addend, and Compare -- find the difference, the child was also asked to write a number sentence prior to solving. Several tasks were also administered to assess each child's ability to use counting-forward procedures.

Each child received nine individual lessons conducted by one or the other of the two investigators. The teaching role was shared by each investigator so that each child had the opportunity to be taught by both investigators. A second adult observer/recorder was present during all lessons. Initial lessons involved an introduction to the computer and production and manipulation of pictorial sets without any accompaniment of appropriate symbolism. Next came the production of sets by means of depression of numerical keys. The third step in the learning sequence dealt with the production of complete closed sentences as representations of solution procedures. The final step involved the "writing" of open sentences with a $\square$ as a representation for the missing quantity. Part of each lesson included a presentation of a variety of verbal addition and subtraction problems to be solved by the student with the aid of the microcomputer. Essentially the student was asked to use the features of the computer program that were currently being taught. On the day following the last lesson, a follow-up individual interview was conducted with each student as a posttest. Six problem tasks were presented with the computer available to assist in sentence writing and solution. An additional six problems with the same semantic structure as the first six with paper/

pencil and physical manipulatives to help with sentence writing and solution.

RESULTS AND DISCUSSION

Prior to instruction, all of the four experimental subjects wrote inappropriate number sentences for all but the most straightforward addition and subtraction problems. Furthermore, they generally viewed the number sentences as unrelated to their solution processes and ignored the sentences they wrote when solving the problem, arriving at a solution by directly modeling the action or relationships described in the problem.

Following instruction all four subjects could write number sentences to represent most problem situations and successfully used this ability to solve a variety of problems using the computer. Three of the four subjects transferred this ability to problems without the computer. To solve a simple word problem, they would first write a number sentence and then use a solution process that modeled the number sentence not the structure of the problem.

One of the factors that significantly facilitated children's ability to represent and solve certain word problems was instruction on writing noncanonical open sentences (e.g., $5 + \square = 13$ and $\square + 5 = 13$). These sentences allow children to write number sentences that are consistent with the semantic structure of missing addend problems. It has been clearly documented that young children solve missing addend problems using an adding on or counting up process which is most closely represented by an open sentence of the form $a + \square = b$ (Carpenter & Moser, 1982). Blume (1981) has demonstrated that children solve these open sentences using the same adding on and counting up procedures that they use to solve missing addend word problems. This body of research strongly suggests that initial instruction on addition and subtraction should include noncanonical sentences. The results of the pilot study strongly support this conclusion.

During and following instruction, all four subjects consistently wrote noncanonical sentences to represent missing addend word problems. In fact, they also wrote sentences like $\square - 4 = 7$ to represent missing minuend problems, even though they received very little instruction in this type of number sentence.

The improvement in representational skills, however, was not totally a function of learning about noncanonical number sentences. Following instruction, the children were also generally more consistent in writing appropriate canonical sentences for a variety of problems and using these sentences as a basis for solving problems. This improvement is most conspicuous for the compare problems. On the screening tasks all four children wrote an incorrect number sentence which they promptly ignored in solving the problem. On the posttest, three of the four wrote a correct canonical subtraction sentence which served as a basis for solving the problem.

Students' performance during the lessons also supports the conclusion that the instruction was successful in developing representational skills and helping the children understand the relationship between their informal strategies and the formal mathematical representations. Children quickly grasped the concepts presented to them and were almost immediately able to use them to solve problems. With regard to the mechanical aspects of the children's interaction with the computer, the results can be characterized unambiguously as positive. All four first graders demonstrated their ability to work with an unfamiliar machine without any difficulty. No mechanical or motor coordination problems were detected.

In conclusion, the computer appears to allow children to rely upon their informal mathematics in an area of formal mathematics such as sentence writing. As we have argued before, the use of verbal problems does seem natural to young children because they are able to solve them in their own informal ways. The present experiment demonstrates that the computer may allow them to represent those problems in a formal way, even though they have not yet completely learned the formal algorithms and number facts. These findings suggest that instruction could be changed to make better use of children's natural ability to solve verbal problems in learning the formal mathematics of addition and subtraction. This pilot investigation suggests that the microcomputer can have an important role in that instruction.

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STUDENT PROGRAMMING IN ELEMENTARY TEACHER EDUCATION IN MATHEMATICS: IMPLICATIONS FOR RESEARCH

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Research indicates that student programming as a tool for the teaching and learning of mathematics is an effective usage of the computer in the classroom. Elementary teachers are faced with the problems of learning how to use student programming in their classrooms. Mathematics education programs for elementary school teachers must be revised to incorporate students' needs for a computing education. This paper will discuss some of the questions that require consideration concerning student programming and the mathematics education of elementary school teachers. Needed research is divided into three categories of topics: (a) student programming as a tool for learning mathematics, (b) student programming as a tool for teaching mathematics, and (c) the place of student programming in the curriculum.

A multi-usage approach to instructional computing features the computer being used in many different ways to strengthen mathematics instruction. Hatfield (1982) characterizes such an approach and discusses implications for teacher education programs. Three categories of computer usages are identified: student programming, computer-based instruction, and teacher utilities. Student programming as a tool for the teaching and learning of mathematics may be the most effective usage of the computer (Luehrmann, 1980; Papert, 1980). Student programming involves the student in activities which require the completing, refining, correcting, writing or executing of a computer program in response to a learning task.

As the microcomputer becomes less expensive, more and more of them will be bought for use in the mathematics classrooms of elementary schools. Elementary school teachers will be faced with the problems of learning how to use them effectively as tools for mathematics instruction. If our elementary school teachers are to develop desirable aptitudes and attitudes for using the computer in their mathematics classrooms, mathematics educators must accept the responsibility for their computing education. Mathematics educators must reevaluate the mathematics education programs for elementary teachers in light of teachers' present needs for training in student programming and the research that has been done on student programming. Previous research has only "scratched the surface" of the issues involved in student programming. This paper will discuss some of the questions that need to be considered that deal with student programming and the mathematics education of elementary school teachers.

It is convenient at this point to divide needed research on student programming into three categories: (a) student programming as a tool for learning mathematics, (b) student programming as a tool for teaching mathematics, and (c) the place of student programming in the curriculum. The first two categories will focus on mathematics students and teachers at all levels. The third category will concern only elementary school teachers; that is, what is the place of student programming in the mathematics education programs for elementary school teachers?

STUDENT PROGRAMMING AS A TOOL FOR LEARNING MATHEMATICS

According to Resnick and Ford (1981), a fundamental assumption of cognitive psychology is that learners construct their own knowledge.

> Learners do not simply add new information to their store of knowledge. Instead, they must connect the new information to already established knowledge structures and construct new relationships among those structures. This process of building new relationships is essential to learning. It means that mathematical knowledge--both the procedural knowledge of how to carry out mathematical manipulations and the conceptual knowledge of mathematical concepts and relationships--is always at least partly "invented" by each individual learner. (Resnick & Ford, 1981, pp. 249-250)

Research indicates that the use of student programming facilitates the construction of mathematical knowledge by the learner (Hatfield & Kieren, 1972; Milner, 1972). By responding to problematic tasks requiring the use of computer programming, students may build up their knowledge of mathematical concepts, procedures, generalizations, or problems. Taking this perspective, mathematical knowledge is not simply passed on from teacher to student as a neatly packaged entity. Rather, through their own activities students construct their own knowledge as they are guided and aided by their teachers.

If students are constructors of their own personal knowledge, then their activity is essential in the mathematics classroom. Piaget (1973) makes this point, "This would also favor the realization [by teachers] of creative vocations in pupils rather than treating them simply as conforming 'receiving' instruments" (p. 79). Piaget stresses the importance of activity with manipulative objects by children as they learn arithmetic and geometry. Student programming can encourage activity by the students. As they respond to programming tasks students become participants rather than spectators in their mathematics classes. According to Piaget (1973), activity and "logico-mathematical experience" are necessary for the development of deductive thought by children. Research needs to be conducted on the role that student programming might have in the development of deductive thought and the encouragement of reflective abstraction by students.

By writing programs students can operationalize and in a sense manipulate their ideas about mathematics. Results of research done by Lankford (Resnick & Ford, 1981) indicate that children's errors reflect misunderstandings of procedures rather than random mistakes. So it is important that these misunderstanding be discovered and corrected by the learner. By engaging in student programming, students can look at and examine their constructions of concepts, procedures, and generalizations. Through this examination they may be able to identify misunderstandings and correct them. Research is required on how programming can help students identify and correct their procedural errors. Since the computer responds to instructions as they are given, it seems that writing and executing programs provide excellent "testing grounds" for students' understanding of a mathematical topic. Student programming may provide a strategic site for research on error analysis in procedural tasks.

From a constructivist point of view mathematical knowledge is built up as students "invent" their own mathematics. Groen and Resnick (1977) investigated young children's inventions of procedures to solve addition problems. Students do engage in creating mathematics, and the role that student programming might have in students' mathematical inventions deserves further attention. What cognitive processes are involved when students invent mathematical procedures via programming?

The effects of student programming on the learning of problem solving require further study. Research indicates that programming may encourage the development of problem-solving abilities (Foster, 1972; Johnson & Harding, 1979). More attention needs to be given to the influence of programming on Polya's (1973) four phases of problem solving: understanding the problem, devising a plan, carrying out the plan, and looking back. Does experience in computer programming affect the ways in which students approach the problems? What kinds of processes do students employ when they use the computer as a tool for problem solving? Polya's fourth phase of problem solving, the looking back phase, is perhaps the phase which is most frequently skipped by solvers. Writing and executing programs provide students with opportunities to look back and consider more efficient or elegant solution paths. Does programming encourage a looking back and reflection on the solution process by the solver? Learning to reflect on their thought processes is very important as students develop their problem-solving abilities and build up their mathematical knowledge

STUDENT PROGRAMMING AS A TOOL FOR TEACHING MATHEMATICS

When the computer is used as an instructional tool in mathematics classrooms, teachers must consider carefully how the computer will be used. The kinds of programming tasks must be considered in light of the content of and goals for the course. Developing experts in programming techniques and languages should not be a primary goal of instruction. Tasks must be conceptualized that will help students develop new knowledge of mathematics and refine and deepen previous knowledge. The tasks should help students discover and correct misunderstandings about mathematical concepts, procedures, generalizations, and problems.

Research is needed on how student programming can be effectively introduced into the mathematics curriculum. Should the programming be integrated throughout the content of the course? The language and techniques of programming could be introduced gradually as they are needed to complete tasks. Should the instruction of the programming language and techniques be given before introduction of the mathematical topics for which programming will be used? Previous research on programming has used both methods: integration into the mathematical content (Washburn, 1969) and separation from the mathematics (Milner, 1972).

Cooney, Davis, and Henderson (1975) describe moves (patterns of language used in the classroom) and strategies (sequences of moves) used by mathematics teachers. Using this framework, what programming moves can be used to teach mathematical topics? What programming strategies are effective in teaching the types of mathematical knowledge? What kinds of tasks will help students build up their knowledge of mathematical procedures, concepts, and generalizations? Tasks aimed at the learning of procedures may focus on the student "teaching" the computer to perform the procedure. To facilitate the learning of mathematical concepts, students may be asked to write programs that will generate examples of a concept or that will test objects to determine whether they are examples or non-examples. Tasks which promote the learning of generalizations may require students to write programs that generate instances of a generalization or to study computer OUTPUT and create their own generalizations.

The computer languages, BASIC and LOGO, are the languages most frequently used in the mathematics classroom. The cognitive requirements for each language need to be studied carefully. The cognitive level at which students begin to learn the programming languages must be considered. What mathematical topics can be taught effectively through student programming? What language should be used when teaching particular mathematical ideas? Are there any difficulties with using both languages in a mathematics classroom?

STUDENT PROGRAMMING IN THE MATHEMATICS EDUCATION PROGRAM FOR ELEMENTARY SCHOOL TEACHERS

Mathematics educators must reevaluate the mathematics education programs for elementary teachers and consider the place of student programming in the curriculum. There are two equally important aspects of the teachers' computing education. The first aspect involves the use of the computer as a tool to aid in the learning of mathematics. In their own college mathematics courses, teachers should experience the uses of the computer as students. Second, the computer should be studied as an object of learning. Pre-service and in-service elementary teachers should learn how to use the computer in their mathematics classrooms.

The content of the mathematics courses for elementary school teachers needs to be reconsidered. The "number-crunching" and graphics capabilities of the computer can allow the teaching of topics and problems that could not be easily taught before.

Research indicates that the use of student programming can improve students' attitudes towards mathematics and mathematics learning (Milner, 1972; Papert, 1980; Washburn, 1969). It is important that prospective elementary school teachers develop positive attitudes towards mathematics and develop confidence in themselves as learners of mathematics. Papert (1980) wrote, "Contact with the computer can open access to knowledge for people, not instrumentally by providing them with processed information, but by challenging some constraining assumptions they make about themselves" (p. 47). The effects of programming on teachers' attitudes towards mathematics, learning, and teaching need to be studied.

The use of student programming may also influence the elementary teachers' views towards their own instruction in their mathematics classrooms. Having experienced programming as a mathematics student, the teacher may have experienced the excitement in learning that programming can lead to. The teacher who has used the computer as a tool for learning may not be afraid to use the computer as a tool for teaching mathematics. The changes effected by programming on their views of mathematics and the teaching and learning of mathematics require careful attention.

In their mathematics methods courses, elementary school teachers should learn about the computer—about how they can use the computer effectively in their mathematics classes. Elementary teachers should understand the role that student programming can play in the classroom and how student programming can help them attain their instructional goals. The pedagogical knowledge concerned with student programming that is considered important for teachers at the elementary level needs to be specified. What activities should be used to train teachers in stategies for using programming in instruction?

CONCLUSION

More research is needed on the effects of the use of student programming in mathematics courses at all levels. Consideration must be given to the goals of the mathematics curriculum and how the use of student programming can help educators attain these goals. The mathematics courses and methods of teaching mathematics courses must be reevaluated in light of elementary schools' teachers needs for knowledge about computers. The teachers must not only learn how to use the computer in their classrooms but also experience the usages of the computer as a student.

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Addendum

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PREFACE

This addendum to the *Proceedings* of the Fourth Annual Meeting of PME-NA contains the three invited papers that were presented during the plenary sessions of the conference. The first paper, by Dr. Kathleen Hart of Chelsea College, begins with the premise that all mathematics education research should be directed toward classroom practice and then goes on to discuss what researchers could and should be doing to help teachers. The second paper, by Dr. Larry Hatfield of the University of Georgia, presents a philosophical framework for guiding the use of computers in the learning and teaching of mathematics. The third paper, by Dr. Stefan Turnau of the Higher School of Education in Cracow, describes a varied sampling of recent research in mathematics education as conducted in Cracow. Taken together, these papers reflect the range of interests, the international perspective, and the concern for improved classroom learning and teaching of mathematics that is uniquely PME.

S. Wagner

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I KNOW WHAT I BELIEVE; DO I BELIEVE WHAT I KNOW?[1]

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"We believe that all these perceptions of the usefulness of mathematics arise from the fact that mathematics provides a means of communication which is powerful, concise and unambiguous" (Committee of Inquiry into the Teaching of Mathematics in Schools, 1982). Thus begins the latest report, the Cockcroft Report, on the state of mathematics in British schools. Needless to say, the picture is not one of pride and satisfaction but more of the "should do better" variety. The report quotes other such exercises from a century, a half-century, and a quarter-century ago, and each in turn has the flavor of "should do better."

The American National Academy of Sciences' *Report of a Convocation* (1982) carries a similar message with the added note: "used to do better." The history of teaching mathematics in Britain fails to support this contention. Prior to 1836 there was no teaching of mathematics at Eton although some arithmetic appeared in the curriculum. At Harrow mathematics was compulsory by 1837, and by 1868 we already have a government report saying. "The teaching of mathematics in English schools is rarely satisfactory." One wonders whether we have ever managed to match the subject to the child.

We do now have some research in mathematics education, but have we used the results of such research to maximum advantage? Is there evidence to show that the results of research have significantly changed practice in schools? Have we in fact been engaged in research that would help the teacher?

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[1] To appear in the Journal for Research in Mathematics Education, March 1983.

THE ROLE OF RESEARCHER

In most Western countries the number of people engaged full time in mathematics education research is very small. The researcher usually exists on grants from external funds of short-term duration, and the expectation of a career in such research is derisory. The research that does exist is to a large extent the result of thesis writing by a number of "apprentice" researchers whose first attempt it is. Our information is therefore fragmentary and very often far from the form in which it can be assimilated by the user (the classroom teacher).

I suggest for consideration the premise that we need so much information, the state of mathematics teaching is so bad, and our teachers are so dispirited that all our efforts in research in mathematics education should be directed to classroom practice. This means that we not only should concentrate on researching issues pertaining to what happens in schools but also should regard it as part of our responsibility to (a) write for teachers, (b) alert teachers in training to the potential sources of information on research, and (c) inform teachers of the limitations, assumptions, and purposes of research.

In teacher training we possibly overemphasize the craft of teaching and underplay the profession so that teachers are unaware of available information. Currently teachers tend to see a piece of research as a total package answering all questions on a particular topic, working from inquiry to solution each time. They need to lower their expectations and with our guidance read research reports with discrimination and an open mind. We need too to help teachers distinguish between research and opinion; too often these are given equal weight. Our opinions tend to be respected because we are "experts"; we should remember that our opinions on mathematics education are probably colored by our prejudices as much as by our other opinions on politics, the state of the nation, or the perfidy of the young.

CHILD METHODS AND INTUITIONS

At Chelsea College we have been talking to children about their mathematics in both the Concepts in Secondary Mathematics and Science (CSMS) Project and the Strategies and Errors in Secondary Mathematics (SESM) Project. The aim of the research is to give information to teachers, so we think it should be almost immediately assimilable by teachers. Our current work on the errors children make is based on the assumption that we can find out a considerable amount about a child's stage of understanding if we study his or her reasoning process when solving a problem correctly and also when using an erroneous strategy (Booth, 1982).

We have taken as our base the data from CSMS in which we had 300 interviews linked to 10,000 written scripts, both sets of data being responses to the same word problems. We think we have described hard and easy types of questions in 11 topics currently taught in British secondary schools, and perhaps more important, we think we know why they are hard or easy. Very often the degree of difficulty is dependent on the availability of naive or *child methods* for the solution to the problem. These methods are not teacher taught, depend largely on whole numbers and halves, and to a large extent involve counting and adding rather than the four operations.

We believe half our secondary school children manage to operate in school without moving outside the set of whole numbers. Many children are therefore receiving lessons on nonintegers for a number of years and yet almost entirely ignore their usefulness. Teachers tell me that they have to repeat the presentation of fractions each year. They also tell me that they themselves use many of these child methods rather than use a formula or rule, yet they deny their own experience and fail to recognize the use of such reasoning in the classes they teach.

A child method is not generalizable. It is never a way of solving all questions of a certain type no matter what complexity of numbers is used but rather a method using simple operations and whole numbers that suffices for much of any exercise the child is required to attempt. For example, in ratio and proportion, when dealing with questions such as that shown in Figure 1, the method used by a large number of our children is additive. Multiplying by 5/2 is not attempted by our average children or for that matter by those whom teachers think are successful at mathematics. Building up to an answer by adding the food for $\underline{x}$ and $\underline{y}$ to find that of $\underline{z}$ is much more popular.

Even when faced with items such as those shown in Figure 2, most of the bright children we interviewed used addition. Ann, whose teacher says

Three other eels, X, Y and Z are fed with fishfingers, the length of the fishfinger depending on the length of the eel.

25 cm long Z

15 cm long Y

10 cm long X

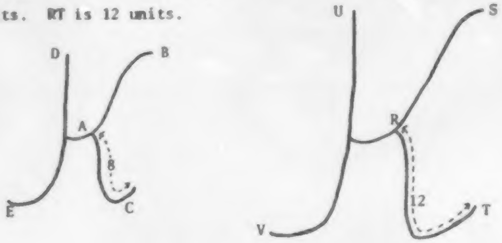
If X has a fishfinger 2 cm long, how long should the fishfinger given to Z be? Z................

If Y has a fishfinger 9 cm long, how long should the fishfinger given to Z be? Z................

If Z has a fishfinger 10 cm long, how long should the fishfingers given to X and Y be? X................ Y................

Figure 1. Item 2b from CSMS test paper on ratio (Hart, 1980, p. 164).

These 2 letters are the same shape; one is larger than the other.
AC is 8 units. RT is 12 units.



The curve AB is 9 units. How long is the curve RS?
The curve UV is 18 units. How long is the curve DE?

Figure 2. Item 7 from CSMS test paper on ratio (Hart, 1980, p. 166).

she is highly successful at mathematics, responded to the first question in Figure 2 thus:

> Ann: Yes, because AC, which is the smaller one, is 8. This curve here, and RT is 12, and AC is . . . RT is half of AC plus AC. So half of AB is 4 1/2, plus 9 is 13 1/2.
>
> Interviewer: Fine. UV is 18, how many is DE?
>
> Ann: There must be 12. Because AC is 2/3 of 12, so in 12 there are three-thirds. And so if that is 18, we need to divide it into thirds, which comes to 6 because 6 threes are 18. Two-thirds there, so that's 12.

About a third of our CSMS sample (n = 2957) gave answers that suggest the use of an incorrect addition strategy as identified by Piaget and Karplus (Karplus, Karplus, Formisano, & Paulsen, 1979; Piaget, Grize, Szeminska, & Vinh Bang, 1968). On the question shown in Figure 2, the children would write that 9 becomes 13. In SESM we have interviewed about 60 children (aged 13 to 16) who give this type of answer. They overwhelmingly use a building-up (or additive) method on questions such as the one shown in Figure 1; they avoid fractions other than one-half; and they appear to regard addition as the operation needed in ratio and proportion problems. Therefore it is not surprising to find that addition is the overwhelming consideration when answering the question in Figure 2 and that the difference between the original line segment and its enlargement is what is added, that is, 8 becomes 12 by adding 4, so 9 becomes 13 by the addition of the same amount. Ann, whose answer has been quoted above, also used addition but crucially realized that the amount added is not fixed but dependent on the original length. The method, however, that has been taught these children is essentially concerned with multiplication. They have adapted that taught method successfully for easy items when multiplication is replaced by repeated addition, but when the question is more complex their need to add overrides other considerations.

The Cockcroft Committee commissioned two pieces of research (Sewell, 1981; University of Bath, 1981) to investigate the mathematical needs of employment and of adult life. Both reports comment on the infrequent use of fractions in everyday life:

> The notation of fractions appears in some clerical and retail jobs, for instance 4 3/7 to represent 4 weeks and 3 days or 2 5/12 to represent 2 dozens and 5 singles. However, school-type manipulation is rarely found and then only in very simple

> cases; for instance, the calculation required to find the charge for 3 days based on a weekly rate is division by 7 followed by multiplication by 3. (Committee of Inquiry into the Teaching of Mathematics in Schools, 1982, p. 22)

These two research reports also note the frequency of nontaught methods being used very efficiently by workers on the job. Indeed, teachers say they use such methods when faced with a need to compute outside the classroom. We have, then, a picture of mathematics lessons in which instruction is taking place but is to a large extent adapted or ignored by the recipient. The teacher bases the exercises set on the use by the children of the teacher-taught algorithms, giving no recognition to the fact that the children do not apply those algorithms.

CHARACTERISTICS OF TEACHERS

Children obtain information by asking questions, but often adults do not or cannot answer those questions. When children start school they still have questions, but as they get older they ask less and supply the missing information themselves. If the supplied information continues unrefuted for some time, it becomes as much part of the children's stored knowledge as what they have been taught. The children we teach are not clean slates on which we write our wisdom; they are continually using their minds to adapt and digest what is around them and so already have many ideas about the world. These ideas may include misconceptions, intuitions, or alternative frameworks; what is clear is that we as teachers largely ignore them and base our teaching on the "clean slate" principle.

In addition, we ignore our own intuitions and frameworks. Teachers of mathematics have to a large extent succeeded in a system where most of their peers have failed. They have succeeded in studying a subject that is unattractive to most of their peers. When we consider the greatest influences upon a teacher's teaching philosophy or style, they are to a large degree the teachers he or she met while a pupil and to a lesser extent his or her colleagues (those who are also the successful survivors of the system). It is no wonder that so many aspects of the philosophy are at variance with demonstrated fact and research results. For example, some teachers say that mathematics is a subject you need not understand initially and that understanding comes after a space of time. When questioned, the teachers support this argument from their own experience, but the examples they quote are usually from fairly high level mathematics and are few in number. Since these examples are based on considerable success in other areas of the subject and an acceptance that mathematics is reasonable, I suggest that the generalization could be invalid when applied to unsuccessful children who consider mathematics to be magic in that it is correct one day and wrong the next.

The belief that it is efficient for a child to be expected to learn an algorithm by rote and reproduce it when needed is another aspect of teachers' philosophy. This belief is often compounded by the added corollary that children who are not understanding the topic sufficiently benefit the most from rote learning, a generalization from the teachers' personal experience that has surely been proved fallacious. Of over a hundred secondary-school-aged children interviewed on ratio and proportion questions in CSMS and SESM, including some highly successful pupils, four have been able to produce a learned algorithm. We have no evidence in this topic of rules learned and repeated without understanding.

Mathematics teachers are accustomed to seeing others fail in mathematics. They very often expect the children they teach to fail in mathematics. There is an assumption that what is taught (even to very young children) is too difficult for some of the learners. Such an assumption is bound to color the attitude of teachers towards their classes; they possibly always aim rather too high and ignore those who need help. We have been trying out some teaching ideas for the remediation of specific errors and have asked teachers to use the designed modules with their classes. It is not unknown for a teacher to say, "They all knew what was required, so I went through it quickly." The pretest (available to the teacher) shows that a number of children are making this specific error, but seemingly their needs are not catered to until some (unspecified) total is reached. Additionally, it is clearly believed that work meant for them would be positively injurious to their more successful peers.

THE CUSTOMER

Teaching is possibly the only profession where the customer is regarded as very nearly always wrong. Judgments are made about how children think and what they want without ever asking them or talking to them. Teachers tell me that they do not have time to listen to a child's description of what he or she is doing even though this appears to be a valuable form of assessment. If in the 11 years of schooling children's opinions on mathematics are never heard, is it any wonder that they consider the subject sterile and impersonal and one to which they personally can make no contribution? If the quotation from the Cockcroft Report that began this paper is true and mathematics is about communication, is it not ironic that mathematics is the subject about which we talk the least?

If we are to improve our teaching of mathematics so that children benefit from it, we must as teachers think again about our priorities. Is it really proved beyond doubt that six lessons a week with 30 pupils is more efficient than three with 15 pupils? Do all children of the same age have to be taught together? Do we really have insufficient time to talk to children about what they are doing? Should we not in fact examine our own prejudices and see whether they are supported by evidence?

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INSTRUCTIONAL COMPUTING IN LEARNING AND TEACHING MATHEMATICS[1]

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Mathematics education is replete with unanswered questions and undebated issues. As an emerging field, mathematics education requires significant scholarly development which is deeply anchored in the most vital forms of creative *instructional practice*. Amidst our efforts to formulate effective applications of instructional computing, it seems paramount to examine the framework upon which we may base our computer usages. The viewpoints posited here do, indeed, represent one personal formulation which may allow the computer to be used effectively in the learning of mathematics. In any event, as it will be asserted in the subsequent discussions, an implied proposition here is the importance of basing our instructional computing upon considered rationales.

Constructivism

Foundational to our conceptions of learning and teaching mathematics are various psychological "theories." Perhaps most mathematics educators adopt elements of several different such theories, a possible consequence of the relative uncertainties of each theory alone serving to explain the complex phenomena associated with our field. A brief characterization of one epistemological framework, constructivism, will be given here as a particularly useful backdrop for instructional computing.

Learning is a person-centered, constructive process. In order for learning to occur, the learner must be an active builder of his own unique, idiosyncratic conceptions. The major purpose of learning is to build-up knowledge. It is both trite but profound to recognize that knowledge cannot be directly transmitted or received. Each one of us must actively engage in mental constructions (and reconstructions) in our efforts to become more knowledgeable. Ultimately then, each person is responsible for his own learning. One role of instruction is to help the learner recognize and accept his own responsibilities to himself for whatever knowledge he is to possess.

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[1] Major portions of this presentation have been taken from the article by the speaker: Instructional computing in mathematics teacher education. *Journal of Research and Development in Education*. (15)4 (Summer 1982): 30-44.

Instruction is a context, as well as a process, for stimulating and guiding each builder in terms of his idiosyncratic constructive processes. The elements of experience, activity, and individuality are presumed to be essential to any learning. Experience includes not only our construction of what happens in a "real world," possibly shared, episode, but also of events which occur in thought, internalized in the mind. Indeed, instruction should aim to facilitate the learner's construction of an ability and a disposition to process reflective abstractions from experiences. By such "looking down" on our experience, and contemplating what it means, we may be able to construct deeper understandings of what we know and how we may come to know.

Activity in learning is essential, for through our actions we deal with situations, both reactively and proactively. Actions here include not only observable behaviors but also internalized mental acts. Through cycles of confronting and responding actions we may construct new ideas and feelings. Lastly, the quality of individuality is inherent in any learning if we accept that we learn from our experience in ways and to extents that reflect the cognitive and affective operations or schemes that we possess at the time. Sometimes we do not learn because we are not yet able to perceive or assimilate elements of the situation we confront. Or we may have a state of feeling which prohibits us from allowing ourselves to benefit from the situation. There may be definite maturational and attitudinal factors in the construction of knowledge which sometime limit us.

Mathematics instruction which seeks to adopt constructivism as a foundational perspective would include approaches that are sensitive to factors of experience, activity, and individuality. It would view mathematics as a complexity of human endeavors from which textbooks and other written records attempt to trace these constructions. To a constructivist mathematics educator, it is a foremost concern to involve the learner in problematic situations and approaches in which direct experience with significant intellectual processing (actions) occurs, and in which explicit attempts at reflectively abstracting about one's processes occur. Thus, a constructivist perspective leads to other frameworks related to human processing.

Processing

The construct referred to as *knowledge* has been analyzed throughout time, and modern psychology and philosophy continue such searches. What does it mean "to know"? In many interpretations a separation is made between two complementary aspects of knowledge: knowing items of knowledge and knowing processes for building and using items of knowledge. Particular items of knowledge of mathematical importance may include solutions to problems, generalizations, concepts, skills, procedures, applications, communications, structures, justifications, feelings, attitudes, values and beliefs. Often such knowledge can be indicated by correctly stating and using particular assertions or explanations within sequences of solving actions. Though these aspects of one's knowledge may appear to be more easily recognized, it is clear that their structures are extremely complex within human thought.

Yet perhaps the other aspect of knowledge is even more elusive. As various items of knowledge are constructed, reconstructed and used, intellective and emotive processes of thinking are also developing and changing. Such processing as abstracting, generalizing, simplifying, patterning, differentiating, particularizing, classifying, guessing, conjecturing, assuming, recalling, concluding, questioning, valuing, and believing may be involved in complex constellations of activity. Without the activation and refinement of the mental operations suggested by these processes one's growth in learning any subject matter would be hampered and limited. It seems sensible that through the use of such processes in building one's knowledge that the processes become more functional, more powerful, and more accessible for subsequent usage. In part we may become more consciously aware of our own processing.

It may be possible to recognize and, to a certain extent, control the use of certain mental processes. As humans, an aspect of our intelligence allows us to reflect upon the ways and means by which we come to know. Thus, by thinking back over the steps followed, the self-questioning used, the decisions made, the reasoning employed, and the results obtained, we may come to understand more clearly what mental tactics we may have followed. Through such reviews of our experiences we may be able to consolidate for future use the essential processes. What was used at this time to build-up a solution to a situation may be strategically helpful in a future task.

How might mathematics instruction foster the student's awareness and application of such processes? Perhaps one emphasis in teaching of knowledge of processes must include the act of analytically reflecting and abstracting upon a learning episode with the learner. By "looking back" upon the possible processing which occurs in constructing an item of knowledge, the student may be helped to become more aware of his own thinking. Of course, labeling a process will not be as important as the activity of attempting to become more aware of what and how one may be thinking at a particular point. Additionally, a second emphasis in teaching may promote knowledge of processes: actually incorporating significant processing in the pedagogical actions used in instruction. In other words the teaching actions employed can embody in a tandem fashion essential qualities of student processes. Teaching processes, such as challenging, questioning, experimenting, conjecturing, assuming, justifying, simplifying, generalizing, observing, modeling, explaining, valuing, judging, expecting, or reviewing, may stimulate and guide students to learn about constructing both aspects of knowledge suggested above.

Problematic Tasks and Projects

Most mathematics educators recognize the importance, indeed the essentiality, of students learning to solve problems. Yet in many cases solving is thought of only as a special, isolated activity. A constructivist, processing-oriented conception of mathematical education would attempt to approach essentially all constructions of knowledge as a problematic situation. Perhaps the following teaching factors may effect the student's growth as a problem solver.

1. *Depth of encounter.* To become a good solver, it seems axiomatic that the student must have intense, firsthand experiences with solving actions. Learning to solve one's problems cannot be a spectator sport! The student must be allowed, indeed expected, to struggle with problematic tasks. This requires a commitment from the student to accept the situation as a problem for him, and to try to persevere in his solving efforts.

2. *Premature modeling.* Many mathematics teachers seem to adopt a habit of demonstrating every step of a model solution to a situation that could be a problem. Students are thereafter only expected to imitate these steps. Socratic questioning is ignored because it takes too much time or effort to help a student to think more independently through his own solution. When a teacher routinely prevents or curtails a student's own solving of a problematic task, the accumulated deficits to a student's potential abilities to solve problematic situations may be overwhelming.

3. *Integrated solving.* In the daily context of building subject matter knowledge it is critical for students to experience a genuine problematic approach. Building-up a new concept, formulating a new generalization, developing a convincing justification or clear description, producing a personalized procedure or algorithm, and constructing a new solution path in an application can be treated as problematic situations. Of course, the key to integrating solving into the construction of all mathematical knowledge rests with the approaches taken in the instruction. Tasks must be set, and situations must be posed, which stimulate and guide such solving.

4. *Thinking about solving.* Many mathematics educators have explored the benefits of emphasizing heuristical advice in teaching (Polya, 1981). Heuristical reasoning would find the solver working on his problem by using plausible but uncertain actions stimulated by questions and suggestions about solving problems. These questions may be either advice provided by the teacher or by the solver himself. Becoming competent to use heuristical advice means that you have various ideas about problem solving, you can recall and use these ideas within the context of trying to solve a problem, and you are ready to learn a new piece of advice or a new outcome for a known piece of advice. In short you can "rise above" your problem-solving activity to reflect upon what you are doing (in order to suggest something to yourself to try) and to reflect back upon what you did (in order to identify processes which may help in solving in the future). Heuristical emphasis in instruction finds the teacher explicitly pointing at ideas about problem solving.

5. *Valuing solving.* Goals of education often cite the importance of developing problem-solving competence. With complex problems facing our civilization we believe in the necessity of preparing coping individuals who

can contribute toward possible solutions. Much of this expectation appears to be given only lip-service in the reality of the classroom. And it will likely continue to be so until educators, parents, citizens, indeed all segments of our society, begin to attribute value to students being and becoming good problem solvers. Teachers must respect and reward themselves and their students for problem-solving emphases. In part what is implicated here is a re-examination of our standards of excellence: we must strive to challenge students with substantive tasks and projects with enough variety to stimulate their differing states of readiness and interest, all the while expecting every student to accept appropriate challenges and strive for excellence in his solutions and productions.

Applying Rationales

In previous discussions the prospect of changing instruction, including teacher roles, was suggested. If instructional computing offers us opportunities to revolutionize education, then we must be prompted to rethink what education might be. In the face of impending change it will be particularly important for us to acknowledge our goals and beliefs, and to consider heretofore unattainable purposes and procedures for educating.

The frameworks briefly examined above may suggest instructional emphases appropriate to classroom computing. With the power and flexibility of the computer students may actively, yet individually, engage in significant processing, including heuristical aspects of problem solving. The constructivist elements of experience, action, and idiosyncracy can be naturally concomitant features of many types of computer usages. Furthermore, explicit attention to processing can be dynamically woven into various computing applications, such as student programming, simulating, gaming, demonstrating, and informing. And through a careful reconstruction of mathematical instruction we may be able to transform our content and goals toward a more routine incorporation of problematic tasks and projects whose solutions are facilitated by students using computers to understand their problem, to construct their solutions, and to communicate their methods and results.

MULTI-USAGE APPROACHES:
A SCENARIO OF INSTRUCTIONAL COMPUTING

To stimulate your visions of what might be possible, let's now consider a brief scenario of applications of classroom microcomputing.

Ms. Terri Walker teaches mathematics at South High School. She is a career teacher with several-years of experience and service, and is engaged in graduate studies in a nearby university. Her high school enrolls students in grades eight through twelve, and Ms. Walker typically assumes a teaching assignment which spans these grade levels because she believes that it is important for her to maintain contacts with a broad curricular range.

At South High instructional computing is a developing emphasis. About four years ago the faculty began a comprehensive planning and implementing process aimed at building toward effective applications of classroom microcomputers. In the period prior, sporadic purchases of microcomputers had led to eclectic, hit-and-miss utilization and effects with few faculty and subject areas involved. Now essentially all curricular areas and all teachers were involved in staff development, experimentation, and curriculum revision in order to develop teaching competencies and instructional materials which employ microcomputers. In the mathematics area Terri and her colleagues have participated in several staff development situations offered by mathematics teacher educators. In one course extensive attention was given to student programming emphases in mathematics instruction; in a second course the teachers examined computer-based instruction and teacher utilities. In a recent course the faculty had learned practical notions of instructional design to be used in authoring and adapting software. Currently, Terri and several other mathematics teachers were involved in a curriculum planning practicum designed to produce instructional computing resource guides to complement the existing mathematics curriculum.

A comprehensive, multi-usage approach has been adopted in developing the mathematics implementations of classroom instructional computing. In this perspective classroom computing is a context of activity for learning and teaching; computers are tools to aid in the construction of knowledge of mathematics. Terri and her colleagues are well aware that any teaching context or aid must be fitted into instruction after careful decisions have been made regarding the educational goals, objectives and subject matter. To avoid the possible distortions to a mature mathematics curriculum which an exciting, eye-catching technology could produce, these teachers first determined what knowledges (i.e., ideas and processes) they would seek to have their students construct. Of course, they tried to be alert to the impacts which computing has upon the discipline of mathematics and upon the myriad of domains of applications of mathematics in society. As they deliberated these matters, Terri was often aware of her need to reconceptualize her own concepts and attitudes of mathematics, per se. Computing and informational technologies were constantly influencing the problems or tasks, as well as the solution processing, which producers and consumers (including her students) of mathematics employ.

Having made a thorough analysis of the aims and contents of their mathematics curriculum, the mathematics teachers were considering the full range of potential usages of the computer in their classrooms. For each topic area they were attempting to identify possible applications, including student programming, eight types of computer-based instruction, and six types of teacher utilities. Similar staff development and curriculum planning were underway throughout the subject matter areas of the high school. One overall goal was that students would experience frequent, varied usages of computers in all sectors of their education, thereby developing competencies and dispositions to use computers and informational technology for personal development and education throughout their lives. Terri was particularly sensitive to curricular distortions which could result from forcing usages where they may not be appropriate; she realized that not all types of applications by either student or teacher would necessarily occur within a particular unit of instruction. In conceptualizing potential computing sessions she tried to assess the possible effects upon the student's learning of mathematics.

The students in Ms. Walker's "foundations of geometry" course were currently studying introductory notions of elementary transformations. The initial experiences involved three primitive motions, slides, flips and turns. Ms. Walker discussed these motions with the class, using a demonstrating program which illustrated through computer-generated graphics various instances of each motion. With the program she and the students were able to produce many cases which were examined with the general question: How can we precisely describe the motion? Eventually, they thought about measuring directly on the TV monitor to find out how each point in the original figure was moved to its image point. Ideas about distance, direction, angles, and arcs led to efforts to pinpoint the simplest prescription for a motion. As always, the effectiveness of the session depended heavily upon the teacher's questioning techniques. After the group discussion, each team of students was assigned to execute the program to obtain more exploration with measures. (Terri often involves three students together in a computer session with one to be "hands-on" person, another to be "checker" and the other to be "questioner" in order to stimulate a good group dynamic for such explorations.) In the subsequent lesson teams had many good ideas to share about how the motions could be described. Out of their discussion came the essential conditions for specifying each motion. At that point, Ms. Walker used a computer-based tutoring program which presented rather concise statements and further examples of how each motion could be produced. It used graphical "rulers" and "protractors" to portray the measures that could be used to dictate a motion. The students were allowed to manipulate and control these electronic "instruments," thus directing the computer to produce a simulated motion. Ms. Walker assigned each student to run a practicing program in which such directions and angles could be given to take some initial shape and subject it to repeated motions, producing an artistic design of the student's choices; these designs could be then electronically colored and either stored for later display or copied onto paper. She also informed the students of a related gaming program which they could use. It allowed two players or teams to earn points by solving various motion puzzles: given a point-set and its image, use the graphics "ruler" and "protractor" to indicate the motion which would yield the image. Scores were awarded based upon how close the player's estimate might be.

Students appeared to be able to find the image if they were given an original point-set and any of the possible specifications of a motion (i.e., a slide arrow, a turn center and arc, or a flip line). Using an interactive simulating program, Ms. Walker engaged the class in a discussion of how one would determine any specification for a motion when you are given a point-set and the image. By "marking" directly on the touch-sensitive screen of the computer montior they tried different approaches with the teacher guiding the exploration through questions and suggestions. Speculations and conjectures could be immediately tested for a variety of configurations which were rapidly generated by the computer upon demand. Soon all students seemed to have formulated sensible *strategies, at which point teams worked together at their own computer to test them for generality. Near the end of the session Ms. Walker distributed several cases which she had produced with a curriculum materials generating utility and a plotter. Some of these examples were set-up to suggest solutions that used compositions of two of the simple motions, a development to be soon approached. Ms. Walker also announced a testing session to be held in three days wherein each student would complete an interactive quiz on the computer, using individually prescribed items from a test generating utility program the teacher had used. As always, the performance results would be saved in a computer file to be accessed and analyzed later by a test scoring and analysis program which entered the appropriate data into files for a grading utility. Terri was hoping that a recently ordered video disk containing short movies of applications of geometry in science and society would arrive. She wanted to use it with its accompanying informing computer programs to allow individual students to search-out uses of the ideas of motion geometry in order to prepare brief written reports; these would, of course, be written and revised by the student using the communicating (word-processing) program.*

Some of the students were having difficulty with orientations of the figures in a flip motion. Ms. Walker asked them to work with her in a session where they first acted on various objects on a table, then used a demonstrating program which permitted simplified, "slow-motion" examples. Several of the students who were particularly keen with computer programming were challenged with several student programming tasks, including the construction of a procedure for generating the graphics for each motion. This led to these students coordinatizing their interpretations of the motions.

Let's briefly reconsider the scenario in light of the perspectives suggested here. Recall that a multi-usage approach was being developed. Within such a comprehensive application of instructional computing a full-range of known usages is considered when planning and conducting instruction. In this way students may come to appreciate the computer as a multi-dimensional tool, and to understand what kind of approach may be productive for particular tasks in learning. As a second feature, recall that the faculty were sensitive to embedding their uses of computers into the goals for student knowledge of mathematics. Though attentive to adjustments in the contents due to computer impacts into the discipline and applications of mathematics, these teachers were sensitive to potential distortions which could result from forcing computing into classroom activities.

In the portrayal of a few lessons involving introductory concepts of geometric transformations several important teaching ideas could be found. Initial encounters with the three geometric motions were treated problematically; instead of the teacher asserting statements of definition the situations of observing instances and describing the motions served to guide students to construct their own ideas. Of course, the teacher's role was to stimulate thinking and to assist in evaluating and refining their ideas. Through such constructivist pedagogy the learner may come to feel that mathematics is a sensible response to a reasonable situation; he may appreciate mathematical activity as a natural human endeavor open to exploration and creation. In the absense of detailed dialogues we can only speculate regarding the explicit attention given to identifying and thinking about processes, yet the context for emphasizing such meta-cognitive aspects appears to be present. Encounters with computers were shown to occur in class-size discussions with the teacher, in three-person working groups and in individual situations. With computer assistance in sponsoring all manner of learning situations it would be possible for the teacher to be freed to attend to various individual encounters with each student; yet to do so must still require the initiative and disposition of the teacher to seek out opportunities to confront the student's thinking. The teacher seems to be aware of student difficulties, however, as she sought out those persons having difficulty with flip motions in order to engage them in extra encounters with the ideas. Additionally, certain students were challenged to construct their own computer programs for completing motions, an enriching development leading to formalization of the concepts.

It must be obvious that the computer in the classroom will offer no panacea, no simple solutions to complex matters of learning and teaching. Yet as a tool available to students and teachers the computer should be used to enhance the activities of each. One implicit, but extremely significant feature, to be suggested by the scenario and by the rationales posited here, must be emphasized: the live mathematics teacher is present and central to the intellectual and emotional development of the student! It may be among the visions of some technocrats to displace the teacher with the computer in the classroom. But the transformations of educating for mathematics which are advocated here place the knowledgeable, sensitive, perceptive adult who enjoys students and the challenge of stimulating and guiding knowledge constructions at the center of the instruction. The computer is a tool for enhancing the encounters which each student may share with this adult related to an activity we may choose to call mathematics. In conceptualizing instructional computing we must keep this image before us.

INSTRUCTIONAL COMPUTING:
PROSPECTS AND ISSUES

Toward Changing Instruction

Perhaps the most basic question for mathematics teachers and mathematics teacher educators due to the potential impacts of instructional computing is: *What might educating be?* That is, given a powerful informational tool through which interactive access to "fingertip" computation, functional analysis, retrievable information, and personalized algorithms is possible, then what kinds of knowledge of mathematics would we seek to have students construct? In particular how might significant processes of thinking be enhanced?

Contemporary emphases in mathematics instruction typically include rather extensive attention to algorithms, i.e., procedures aimed at "knowing how to." Usually only a single, inflexible, "ready made" procedure is presented. The steps of the procedure are described or exemplified by the textbook, and these are typically "modeled" by the teacher's actions in discussions of specific examples of appropriate algorithmic tasks. Student learning of "the" procedure is often expected to be imitative behavior. Little or no attention is given to building-up algorithms by each student as possible solutions to a problematically treated algorithmic task. Students are often neither encouraged nor allowed to offer alternative procedures, or to adapt (i.e., personalize) procedures. However, computer programs are often thought of as algorithms. The construction of a computer algorithm can involve many important cognitive processes, as well as emphasize understandings of the mathematical ideas needed to build-up, test, "debug" and refine the student's own procedure. Interactive computer programming can allow the student to experience practical heuristical advice as well as concrete notions of algorithmics. Thus, we may note one important question related to student programming usages of the computer: *How can our teaching of mathematical algorithms be enhanced through student programming emphases?*

Several additional issues are embodied in student programming in the mathematics classroom. During the past two decades, mathematics teachers have recognized the almost natural linkages that the activity of computer programming appears to have with mathematical problem solving. It has been a typical response to include a computer programming course in the upper high school mathematics curriculum; more recently these offerings have been labeled "computer science." Herein we must now express a concern: *Why should a mathematics curriculum include "computer programming" or "computer science" courses, per se?* If the goal is to develop skills of programming and computing, as an end, then the mathematics curriculum is being distorted. To be sure, many computer programming courses draw upon mathematical situations, and may thereby reinforce and enhance the student's knowledge of mathematics. But if such mathematical situations are simply serving as settings to learn a computer programming language or computer science concepts, then these courses misrepresent the goals of educating for mathematics. The point of view to be advocated here is that student programming usages should be embedded as problematic algorithmic tasks reflecting a careful conceptualization of mathematics; the mathematics to be learned should determine the computer programming emphases, and not vice versa. Therefore, specialized computer courses should be placed within other components of the school curriculum, such as vocational education or science education.

Yet another type of isolated, specialized curricular offering has made its appearance in the school curriculum: the "computer literacy" course. Considerable debate continues relative to the question: *How can we develop computer literacy in our societies?* Often the mathematics program is implicated as the setting for a typically brief, awareness orientation to computing in our society. Though such treatments may be beneficial, it must be realized that to become "literate" with computing, a student must experience the machine and its applications throughout one's schooling. Through a comprehensive application of instructional computing we can offer the wide range of encounters with the machine that would foster deep understandings of the capabilities and limitations of the technology. By using the computer for student programming as well as all forms of computer-based instruction, the student can experience computing in relation to knowledge of the disciplines upon which school curricula are based. Any approach which places "computer literacy" as a special, short-term course apart from studies of mathematics, sciences, languages, etc., must surely be short-sighted and ineffective in promoting actual literacy.

Lurking throughout any considerations of the changes which computing may allow in our mathematics instruction is the question: *What can the roles of the mathematics teacher now become within a comprehensive usage of the computer for instruction?* Indeed, if the potential benefits of instructional computing are to be realized, then the part to be played by the mathematics teacher must be thoughtfully reconceptualized. To some technocrats, computers in the classroom should be developed to "teacher-proof" the curriculum; in this sense the teacher would be essentially eliminated from significant participations, with the machine sponsoring the critical instructional events. This must surely be an erroneous approach. Rather, we must place the mathematics teacher centrally as the essential element of decision-making in the instruction. With the computer as an instructional tool, the human teacher should be ready to capitalize upon deep, individual interactions with students. The teacher must become more focussed upon the stimulation, facilitation, and interpretation of the student's complex processing we call "learning." With viable computing activities available for students to engage, the teacher must be able to help students encounter those activities appropriate to their learning; thus, the teacher must be ready to direct students to various student programming and computer-based instruction situations. In some cases the teacher must be able to prescribe to a computer program those parameters needed to generate the session deemed appropriate for a particular student; thus, the teacher will need to be knowledgeable of the instructional package available on the computer. With considerably more detailed information possibly available regarding a student's learning "history," this teacher may need to know how to access and interpret such information from stored files; thus the teacher will need to be disposed to think deeply about the possible relationships between intellectual tasks and performance indicators, and how such relationships might be explored further through individual interviews with the student.

Many possible implications for the teacher's roles could be cited. With effective computer-based practicing sessions available, a teacher may be able to direct attention to creating and sponsoring other, higher-order activities, such as applying mathematics. With powerful computer-based simulating programs available, the mathematics teacher may need to use a pedagogy that is much more exploratory than one's traditional expository approaches. Or with a significant orientation toward student programming, the teacher may need to shift attention to the setting of an algorithmic task as a problematic, constructive situation rather than the traditional modeling/imitating paradigm. And with an effective usage of computer-based communicating wherein the mathematics student can construct communiques, a shift toward engaging students in comprehensive problem-solving projects may lead to altered roles for the teacher.

Up to this point I have highlighted mathematics instructional computing in terms of potential types of usages and some related issues. We have been assuming a rather unchanging body of mathematical contents, focussing upon the ways our mathematics instruction might be changed. Taken as a tool for instruction, the computer could clearly be used to foster significant constructions of the contemporary mathematical knowledges emphasized in today's curricula. But what about the computer's potential impacts upon the nature of mathematics, per se?

Toward Changing Mathematics

Historically, computing technology has evolved through significant mathematical developments. To a lesser extent, perhaps, increasingly powerful computing technologies have altered the conduct of mathematics. To be sure, in many domains of mathematical research the availability of the high speed, digital computer has served to stimulate as well as facilitate new mathematical breakthroughs. Yet the conceptions which mathematics educators from preschool through college have held regarding the nature of the discipline of mathematics and its learning have typically not been influenced by modern computing tools. Many indicators of this failure to impact upon mathematics curricula can be found. Though calculators have become increasingly ubiquitous, offering powerful inexpensive computational capabilities to essentially all citizens, we still find traditional emphases on paper-and-pencil calculations throughout all levels of mathematical education. Indeed, many mathematics teachers not only ignore such computational assistance in dealing with mathematical tasks, but prohibit their students from employing calculational tools in their solutions.

The prospects for inexpensive microcomputers impacting upon the "doing" of mathematics are great. The issues involved in allowing such impacts to evolve are also numerous, and in many respects are not new. For brevity let us focus upon one basic question: *What is the nature of mathematics as it relates to essential goals for mathematics education?* It seems that to many educators mathematics is a body of information, existing "out there" apart from human minds, to be somehow internalized through interactions of teaching and learning to be found in classrooms. In this perspective mathematics is to found "in the textbook" or journal article, and a major task for teacher and learner is to retrieve and store (i.e., remember) this information. To those who view mathematics externalized, the emphasis is often upon "finished" items, such as statements of definitions or theorems, completed written proofs or solutions, and recalled steps of a procedure or rules of a prescription. Teaching is often construed as informing, and learning is characterized as information processing.

An alternate view of mathematics embodies a different conception of reality and epistemology (e.g., Plunkett, 1981). In this perspective mathematics, per se, has no existence apart from human intellects. Any discipline exists only as a complexity of human endeavors and human constructions. Written and spoken communications attempt to record and trace these constructions. However, mathematics, as a way of thinking about, or processing, a great range of problematic situations experienced by people, cannot be adequately recorded. It must be constructed and reconstructed by each person. Thus, the notion of "teaching mathematics" focusses upon the stimulation and guidance of a human mind with respect to building and rebuilding certain ideas; attention would also be given to thinking about one's processes for constructing ideas. To be sure, there is an aspect of this conception of mathematics education which involves social transmission: the teacher draws upon personal experience (i.e., one's prior experiences of constructing your own knowledge) as well as source of information (e.g., textbooks, advice of others). But developing one's own knowledge of mathematics is not accomplished simply by such transmissions, but rather depends upon each individual constructive experience. Building-up mathematics in this sense can only be accomplished through constructive encounters with significant problems (i.e., nonrountine challenges requiring solving actions). Thus, "doing mathematics" is not portrayed by imitative behavior of students or modeling actions of teachers, but requires individual initiatives in building cognitions unique to each learner. The factors of personal experience, activity, and individuality become essential to such mathematics instruction. And a significant dimension of teaching centers upon an effort to stimulate certain cognitive processing and to attempt to understand the processing being used by the learner during his construction. Helping the student to reflect upon his own thinking can bring an important heuristical emphasis into the situation.

There may be differing consequences for the possible roles of microcomputers in mathematics from these contrasting viewpoints of the nature of mathematics. In the more realist orientation it becomes paramount to learn traditional ideas to be found in published mathematics: for example, it would be expected that one would somehow come to know *the* steps of a classical proof or algorithm. In the more constructivist perspective of mathematics it is less crucial to repeat the building-up of such traditional solutions or approaches: confronting various (perhaps classical, at times) situations should lead to a variety of solution paths, with an emphasis upon generating and refining one's own results (though they be at times fraught with errors and inefficiencies, and lack the elegance or precision of other approaches).

Thus, in the more realist perspective of mathematics, instructional computing would be focussed upon the interactive transmissions of information. Various types of computer usages, such as practicing, tutoring, simulating, or demonstrating, would be used to feature concepts, generalizations, or skills as particular statements or rules to be somehow assimilated. "Doing mathematics" within such instructional computing largely represents comprehension of presented information. Classical notions, such as "the long division algorithm" or "solving a system of two equations in two variables by the substitution method," are retained in the teaching or mathematics, though they are essentially archaic developments in the face of alternative numerical or analytical procedures possible through computing.

In the more constructivist orientation toward mathematics one might find the potential nature of instructional computing to be very different. In the context of treating more classical situations the emphasis may be upon setting as a task the building of a procedure or the construction of a generalization. Singular, monolithic responses are replaced by diverse alternatives, both in terms of the possible productions as well as the paths and means followed in getting to an end result. The computational and analytical capabilities available on the computer are taken as permissible functions and operations to be used in solving tasks aimed at understandings of fundamental ideas and processes. Thus, the computer is acceptable as a calculator; furthermore, recent utilities (see esp. Wilf, 1982) which allow the microcomputer to perform analytical transformations (e.g., algebraically solve an equation, evaluate algebraically an integral, or simplify expressions) would be accessible for students to use. A greater emphasis would be upon knowing why and when to use particular computations or manipulations, with concomitant de-emphasis upon being able to actually perform such procedures. Additionally, such computing capabilities would allow the stimulation of significant problems to be solved, hypotheses to be tested, or assertions to be confirmed, both as they may be formulated by the student or suggested by the teacher.

These contrasting perspectives regarding the nature of mathematics and its "doing" suggest rather differing implications for instructional computing in mathematical education. Of course, an apparent dichotomy such as these points-of-view appear to represent is likely to be a false representation of the circumstances of teaching mathematics with classroom computers available. Almost certainly students will need to engage in informational processing at times; even radical constructivism must realize that not all of one's mathematical knowledge will require personal invention. And throughout a student's experiences in expository, information disseminations, considerable intellectual activity occurs aimed at reconstructing the "presented" mathematics for oneself. Indeed, there can be considerable blurring of these two epistemological frameworks in the actual classroom. However, considerable restraint, indeed prohibition, can be exerted upon the student due to the conceptions of mathematics we hold. In the face of powerful numerical as well as algebraic tools we must now thoughtfully consider our notions of what consititutes mathematical knowledge and its use.

PREPARING MATHEMATICS TEACHERS FOR INSTRUCTIONAL COMPUTING

What approaches might be taken to prepare mathematics teachers for instructional computing?

1. *To know instructional computing deeply and vitally, the teacher trainees must experience the computer within their own role as a student.* This perspective implies that the teacher must actually study mathematics in which his own constructions of knowledge involve various types of computing. This suggests that the integration of instructional computing into mathematics courses must be undertaken. Or within professional courses, such as the study of curriculum or of pedagogy, the teacher must experience treatments of at least some portion of mathematics in which a rich, varied utilization of the computer is functioning.

By exploring new concepts and generalizations, by solving significant problems, and by practicing new skills which they themselves are engaged in constructing, the teacher's should come to understand the potential influences which the computer in instruction may have. The teacher will experience the learning of computer programming within the construction of her own procedure for solving an algorithmic task. The teacher would witness the teacher educator using computer programs for demonstrating, simulating, gaming, testing, etc. Thus, not only will the teacher come to know instructional computing from the perspective of a learner, but will be able to reflect upon the nature of the instructional environment in which their teacher, the teacher educator, functioned.

2. *Specific attention to building didactical knowledge for instructional computing must be given.* What is sought here is explicit attention to the psychology, philosophy, sociology and pedagogy of conceptualizing and conducting classroom episodes in which instructional computing sessions occur. An approach may be to incorporate treatments into extant professionalized education courses, such as foundations, educational psychology, curriculum and methods of teaching. In this integrated approach the computer is treated as a particular technological tool along with others. A different, or companion, approach would be to feature separate instructional computing courses. One curriculum might begin with a course on algorithmics and student programming, dealing with the curricular and methodological emphasis of constructing computer programs. A second course would introduce computer-based instruction, its characteristics and usages with particular attention to designing and conducting teaching episodes in which prepared computer programs for instruction are employed. A third course would help the teacher to develop sufficient computer expertise and instructional design knowledge and skill to author original computer-based instructional programs; attention would also be given to adapting or modifying existing programs. Throughout all such specialized courses concern for the practical, yet thoughtful, implementation of multi-usage approaches embedded in sound curricular decisions would be central.

3. *The mathematics teacher must experience the impacts of the computer into the "doing" of mathematics as it is now affected by powerful computational and analytical procedures.* It has become clear that modern calculational devices will receive little application into mathematics instruction without at least teachers understanding the changing nature of mathematical activity. It must now become a high priority for mathematical teacher education to assist teachers in understanding the situations in which sensible emphasis upon performing

procedures without machine assistance may occur. That is, it should now become our attitude that a requirement to perform routines which can be done by machine must be justified; students would not be expected to perform such routines unless it can be shown that such mental or paper-and-pencil processing is an essential element of one's mathematical knowledge. However, with such a posture regarding traditional algorithmic learning it will be incumbent upon us to develop deep meanings for mathematical operations and functions in order to allow the student to experience the personal development of problem-solving competence toward which we now may direct our attention. In our past attention to skill in calculating we may have provided needed experiences in using the ideas in ways which deepened the student's understandings; with greater emphasis upon knowing why and when to use machine procedures we must focus even more concern upon sound understandings of the underlying operations and functions.

4. *Comprehensive instructional computing will involve teachers in all subject matters of schooling across all grade levels, and mathematics teacher education must reflect such comprehensive developments.* In schools where computers are being introduced it is often the case that the mathematics teacher is among the initial innovators. Indeed, many mathematics teachers are called upon to provide leadership within the school's efforts which may involve them in conducting local staff development and in chairing local curriculum committees. Comparable leaderships may be imminent for many mathematics teacher educators within the colleges of education. In part such involvement may be a natural result of the perception that the computer is, after all, a calculator and that the business of computer programming has long been engaged in by mathematical types. If we are to promote educational goals that result in significant computer literacy, that free the learner to know how to use the computer for life-long self-educating, that advocate a deep transformation of education from mindless information processing toward constructive problem solving, then we must become involved with all curricular areas of schools. Such potential impacts of instructional computing will require a comprehensive approach. And we must be willing to reconceptualize teacher education in ways which recognize the general as well as subject-matter specific applications of computers in instruction.

The effects of educating mathematics teachers for instructional computing according to the approaches outlined above await investigation. What is sought is to allow a teacher to develop knowledge and appropriate skill to be able and willing to apply mathematics instructional computing in their classroom. Teachers should be able to identify and set up situations in which students could build, test, correct and refine their own algorithms (i.e., computer programs). Teachers should be able to incorporate various computer-based instruction programs into the student activities. With each type of computer usage, such as demonstrating or simulating, the teacher will need to learn how to capitalize upon the computing activity as an enhancement of mathematical learning. Teachers should be able to employ teacher utilities in aiding their efforts in preparing materials, evaluating learning, and managing instruction. Teachers should be able to alter or adapt particular computer programs to the applications they need for individual students and local curricula; such competence will require mature computer programming knowledges, and sources of software will need to allow teachers to make such adaptations. As with any tool or approach, teachers will need to be able and willing to evaluate potential applications as well as attempted applications. This attitude of evaluation must extend to a professional review of software, of teaching strategies, of learning effects, and of self-evaluation. Finally, teachers must be able to keep abreast of the rapid developments of hardware and software for instructional computing which appear to be a part of this technological "future shock."

Few clear strategies for building appropriate mathematics teacher education programs involving instructional computing are apparent. Certainly one central concern relates to the immediate, pressing need to prepare teacher educators in all areas to use computing technologies with teachers. Though potential uses of computers for instruction have been explored by isolated enthusiasts for over two decades, few teacher educators have yet become involved in instructional computing. And without such competencies and dispositions it seems unlikely that any further implementations and developments will transpire. Sporadic, narrowly focussed impacts are likely to occur in the absence of visionary, comprehensive considerations, with corresponding nominal effects upon the lives of teachers and their students in schools. And it may be equally risky to view preparation of the teacher educator as a simple matter of a few staff development sessions which allow superficial awareness without deep experiences in understanding what it feels like or means to learn with the computer as a tool.

Indeed, we confront a classical "chicken and egg" or "bootstrap" problem. For the current generation of teachers, and teacher educators, have not been allowed the opportunities to know what the building of mathematical knowledge may be like wherein the computer serves as a powerful tool and context for learning. Yet, somehow we must respond to this rapidly developing technology, and we must begin immediately. To delay further may certainly foreshadow the alteration of education from other sources, such as the profit incentives of massive sales of self-contained instructional packages to be used apart from the school. Perhaps our only means is to engage in bona fide problem solving: we can adopt the development and research of instructional computing at all levels of mathematical education as our highest priority (see Recommendation 3, NCTM, 1980). By engaging in a period of exploration and reconstruction, we may expect teachers and teacher educators alike to create and modify many forms of computing in the mathematics classroom. Significant resources must be brought to bear upon such creations and their sharing. Throughout we must be willing to reconceptualize our most basic premises and traditions for educating in order to allow our students to derive the benefits possible from this technology. At the same time, we must confront the potential impacts of computing in ways that help us to realize and clarify our beliefs about what educating may be. Perhaps as we enter the twenty-first century we will understand more clearly what symbiotic relationships with computing technology will allow teachers to enhance the building-up of mathematics with their students.

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MATHEMATICS EDUCATION RESEARCH IN CRACOW:
SELECTED ISSUES AND RESULTS

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Higher School of Education (WSP) in Cracow (one of seven in Poland) is the main Polish center of mathematics education. It is probably best described as a teacher training university. That is, its departments correspond to most school subjects and its primary mission is to train teachers of all school levels, but its academic program is that of a university. The WSP has parallels in other countries, e.g., the Paedagogische Hochschule in Germany. Each department in the WSP is called an institute and includes a subdepartment of education for the corresponding subject. Thus the Department of Mathematics Education comprises part of the Institute of Mathematics, and is therefore closely linked to mathematics but is separated from psychology and pedagogy, which are in another domain. The Department of Mathematics Education was established in 1958 with three faculty members; now it includes more than fifteen professors and instructors, most of them former graduates of WSP.

The mathematics education research that is conducted in this department (see Turnau, in press) has been strongly influenced by the personality and views of Anna Sophia Krygowska, its founder and head until 1972, now professor emerita. Her position with respect to mathematics and its teaching will now be sketched as a background for the examples of research that will follow.

KRYGOWSKA'S PHILOSOPHY

According to Krygowska, mathematics is the study of abstract structures. This definition applies equally to mathematics research and all levels of learning. It is only the complexity of structures, the teaching methods, and the motivation that vary — the essence of learning mathematics is always the same, regardless of the level. So there is mathematics in a child's sketching of the street plan of a town to show that he can get home without crossing the main street, whereas drawing a right angle triangle and measuring its sides to verify the Pythagorean relationship is not a mathematical activity. The former involves reasoning about abstract geometrical notions supported by a paper-and-pencil sketch, while the latter is purely an experimental investigation of a physical object.

Because mathematical structures are abstract, the only access to them is through language (verbal, symbolic, pictorial, or other) and reasoning. Therefore, any abstract notion that is to be handled in a mathematical way must first be properly defined. Imagination, intuition, and the like are helpful, but the ultimate goal is mathematical discourse. Even visual imagination in geometry is not simply the ability to "see with the eyes closed," but it also includes the ability to reason about the objects that are "seen" as well as to "manipulate" these objects accordingly.

Abstract structures of mathematics, in order to be studied and eventually applied, must first be acquired or reconstructed by the individual. These structures are accessible to the average human being because of their dynamic character: Mathematical activity consists in performing mental operations on abstract objects. Learning mathematics is acquiring these operations and the skill of using them purposefully and efficiently. (Here, the word *operation* should be understood in its broadest meaning, not only in the algebraic sense.) Krygowska describes this process using the Piagetian term *internalization* and identifying its three stages, from concrete manipulations through imaginative manipulations to abstract operations. The task of the teacher is to provide appropriate material and orient students' activity toward internalization of certain mathematical operations. This task remains essentially the same whether it is a lesson on counting numbers in grade one, real numbers in grade eight, proofs in grade ten, or integrals in grade twelve; it is only the "material," language, and motivation that have to be adjusted from lesson to lesson.

Mathematics, being the most abstract of all sciences, is the most suitable subject for developing the ability and taste for a theoretical approach to real and practical problems--the very human way of solving them. It is only for this goal that mathematics is worth teaching to everybody. And it *is* worth teaching.

It should be noted that the above is a sketch of my own perception of Krygowska's "credo," internalized during more than twenty years of learning and working under her leadership. Unfortunately, her philosophy cannot be found in a concise form in her publications. It is scattered throughout her *Zarys* (1977), available only in Polish as yet.

SOME RECENT RESEARCH

Two of the four studies that will be described (all completed at WSP during the years 1971-1981) concern reasoning and the use of language in the learning of mathematics. The other two consist of construction, implementation, and evaluation of a teaching project. Each of the four concerns a different school level, each approaches the collection of data differently, and each analyzes the data differently. So this small sample seems to illustrate very well the Cracow style of studying mathematics education problems.

B. Nowecki, *A study on the effectiveness of building up the notions of theorem and proof in high school mathematics* (Dissertation for the habilitation or second degree doctorate).

The following two research problems were posed:

1. To what extent does the teaching of mathematics in high school result in students' understanding of the basic methodological concepts of mathematics (like theorem and proof)?
2. Does the introduction of the elements of logic and set theory into the high school curriculum reduce students' difficulties in the understanding of:
    a) the hypothetical structure of a theorem,
    b) the proof of a theorem,
    c) the application of a theorem, in particular, based on the modus ponens principle.

A total of almost 700 students in grades 10 to 12 and high school graduates were tested with the help of problems like the example below.

> PROBLEM 1. Read the theorem below, examine the proof, and answer the questions that follow.
>
> THEOREM: If any even number greater than 2 is the sum of two prime numbers, then any odd number greater than 7 is the sum of three prime numbers. [A detailed proof is given.]

QUESTIONS:

1. Do you accept the proof as correct?
2. Do you think that the theorem is true?
3. Can you draw from this theorem the conclusion that any odd number greater than 7 is the sum of three prime numbers?

Justify your answers.

After answers to the problem above were collected, students were given the following:

> PROBLEM 1′. In the theorem of Problem 1 the truth values of both the premise and the conclusion are not known. The statement that *any even number greater than 2 is the sum of two prime numbers* is known as Goldbach's conjecture. Until now it has been neither proved nor refuted. Thus we do not know whether the premise of the theorem is true or false. And, it has never been decided whether the conclusion that any odd number greater than 7 is the sum of three prime numbers is true or false. With this information in mind, answer again the questions [above].

It should be noted that in the Polish high school (grades 9 to 12) geometry is taught throughout the four years, occupying about one half of the mathematics curriculum. Emphasis on precise language and proof was even increased by the reform of 1968 which introduced a concise course on elementary logic as part of the mathematics curriculum in grade 10. Nowecki's research lasted several years, during both the pre- and the post-reform period. Though this dissertation only discusses the post-reform research results, the supposed impact of the research is implicit.

Astoundingly large numbers of the subjects tested by Nowecki (precise data are not at my disposal) responded incorrectly to questions 2 and 3 in both cases: "yes" and "yes" in Problem 1, "no" and "no" in Problem 1′, thus clearly associating the truth value of the theorem with the truth value of the premise and/or conclusion
and not with its proof, and also identifying the truth value of the conclusion with the truth value of the theorem. Similar results were found in connection with other test problems, despite variations of the mathematical content and other features. The author concludes:

> Students by and large do not know the basic rules of logic, indispensable in ordinary mathematical reasoning. The most striking example is the ignorance of modus ponens. What is most alarming is not the lack of knowledge but the lack of understanding of the subject matter. Without exaggeration, it can be concluded that the majority of the notions, laws, and rules of logic are entirely inaccessible to the students. They escape students' intuition and students' ability to handle them and apply them properly. And yet these notions, laws, and rules are included in the curriculum and taught to the students. The lack of understanding of the formal sense of truth of a theorem is widespread among students. In deciding the truth of a theorem, they let themselves be guided by intuition based on practical or experimental verification, referring to selected numbers or figures. This intuitive conception of the truth of a theorem is so deeply rooted that the students judge a theorem according to their practical methods even when they have acknowledged the correctness of its proof. [My translation]

All this entirely escapes our attention if only standard mathematical problems are applied as test problems - hence the surprise. But it also seems to escape the learner's attention. One can suspect that such misconceptions persist in many students throughout the college years and - horrors! - also persist in many mathematics teachers. (This would be an interesting, though embarrassing, research question.) If this is true, are we deceiving ourselves that we teach mathematics, or rather is it the case that school mathematics is, and must be, a subject essentially different from the science of mathematics? Well, there is a third alternative: That is, our teaching of mathematics does not engender genuine understanding of the deductive method. It is the teaching that should be changed, not the mathematics or the students.

H. Pieprzyk, *Assessing reductive reasoning in pupils in grades two to five and developing this reasoning with the use of didactic games* (Unpublished doctoral dissertation).

Reductive reasoning (or *reduction*) consists in seeking conditions sufficient for the conclusion (to solve a "prove" problem) or finding the unknown number, figure, etc. (to solve a "find" problem). Reduction should not be confused with the "working backward" strategy or reversed reasoning, in which the conclusion or result is assumed as a premise and the original premises are derived in the hope that the logical derivation will be reversible.

Reduction is a powerful and highly effective strategy in solving many problems and proving theorems. Surprisingly little attention has been paid to it in the research on mathematics problem solving. It is doubtful that many students come up with it spontaneously, as they do the indirect proof strategy. Reductive reasoning requires special training, long enough to develop not only the necessary skill but above all the will to apply it voluntarily where appropriate. When is the earliest time and how should this training begin were Pieprzyk's research questions. After ascertaining that reduction is too difficult for pupils in grades two and three, she undertook a teaching experiment in grade four that aimed at developing this kind of reasoning. As vehicles, special games and word problems were used. Here are three examples of games:

"Race of the Pawn" Game: Two players take turns moving a pawn on a grid as on Figure 1. Whoever places the pawn on the goal wins.

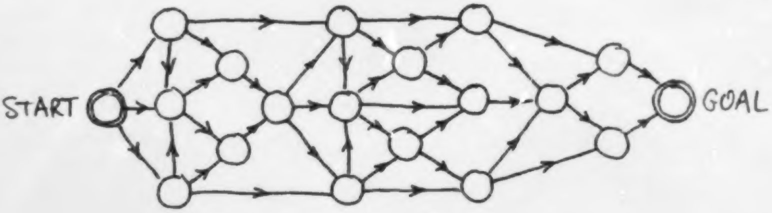

Figure 1. "Race of the Pawn" game grid

"Race to the Number N" Game: Starting with 0, two players take turns adding numbers less than K to reach a goal number N. Whoever reaches the sum N is the winner.

"Race on N Pillarets" Game: Players take turns crossing out a number less than K of a row of N parallel strokes. The winner crosses out the last stroke. This is a variation of the well known game "NIM" in which players pick up matches.

The word problems, both in teaching and in the tests, included not only standard fourth grade problems but also a few specially designed ones, in which reduction was particularly pertinent or even an obtrusive strategy.

Though certain written materials were collected and analysed (for example, children played the "Race on Pillarets" game using pencils of different colors, which allowed their strategies to be analysed), the main method of this investigation was an informal observation of children's classroom activity and talking to them both in and outside the class. Instances of reductive reasoning or its blockage were recorded. To answer the question of transfer to ordinary word problems, problem solving sessions were observed in two classes - one in which experimental teaching of reduction had been conducted and one control class.

From such "circumstantial" evidence the following conclusions were drawn:

1. Fourth grade students do not spontaneously apply reductive reasoning unless they are taught to do so;
2. Fourth grade students can successfully be taught to apply reduction independently in their search for the winning strategy of certain games;
3. There is a transfer of the ability to apply reduction in games to applying it in mathematics problem solving.
4. Consistent application of the reductive approach to solving word problems in grade four induces students to apply this method independently to solve such problems.

The formulation of conclusions must be extremely cautious, in view of the research methods applied and the evidence collected. It would probably require a years-long experiment for one to be able to infer positive conclusions and implications for teaching. Nevertheless Pieprzyk has shown that early introduction of children to reductive reasoning does make sense and that games, a highly motivating device at early ages, can be used for this purpose. This result seems to be educationally significant.

M. Sznajder, *Texts guiding students' work in mathematics teaching in middle grades* (Unpublished doctoral dissertation).

The main educational goal of this project was attaining maximum involvement of all students, particularly those of average and poor classroom performance. For this purpose student work sheets were prepared in the French "fiches de travail" semiprogrammed style, with both "horizontal" (i.e., relative to the student's estimated ability) and "vertical" (i.e., relative to the student's actual performance) dimensions. The topic selected for the experiment was the elements of probability. Experimental teaching was conducted both in one eighth grade class and with a small laboratory group of students from another class. An eighth grade control class was taught the same material in the traditional style.

The project was evaluated mainly through direct observation of students' work as well as through analysis of completed work sheets and achievement tests given in both classes. But another approach was used to evaluate students' heuristic attitudes and skills: probabilistic games, which the students had encountered for the first time. In particular, the students' active attempts and ability to apply their knowledge in the new problematic situation were assessed.

Achievements of the experimental class were higher in all observed aspects. It is worth emphasizing that the project proved to be most beneficial for average and weak students, who are the object of Krygowska's main concern and hence of most researchers working under her leadership.

G. Trelinski, *Management of the calculus class through guiding students' individual preparation* (Unpublished doctoral dissertation).

This project was devised for the teaching of the first calculus course in the Higher School of Education in Cracow, a course that is a persistent headache, since brilliance in mathematics in a teacher training institution is rare. The teaching there, as in most European universities, is strictly divided into lectures, given by professors to large groups of students, and practice classes, led by instructors in small groups. Trelinski was an instructor who worked in close cooperation with the professor whose course was attended by all of the students in Trelinski's two classes (experimental and control). Students from these classes were matched according to certain criteria and thus two equivalent groups of 21 students each were formed.

The main feature of the project (implemented in the experimental class) was written material that the students used when preparing at home for practice classes. This material aimed at guiding and organizing the student's learning rather than directly teaching him the matter covered in the last lecture. Though formally it was programmed material, it was not at all like standard programmed instruction. For example, the solutions of initial examples were partitioned into rather small units, but as learning progressed programming was more and more "coarse," ending up with plain problems to be solved without help. The material often referred the learner to textbooks and other sources, thus helping him to acquire the skill of studying mathematics independently. Lessons were mostly spent discussing difficulties and other issues proposed by the students. Periodically tests were administered.

Students of the control group did not receive the project material but instead exercises were assigned whose solutions were presented in the class by selected students. The periodic tests were the same in both groups.

Effects of the experiment surprised Trelinski himself. Here are a few of the most striking numbers: The dropout rate was 15% in the experimental class and 35% in the control class. Test scores averaged 25% higher in the experimental class than in the control class, even though the groups were matched (significant at the .05 level). The most dramatic discrepancy between the two groups emerged when solutions to problems were compared, problems which were technically easy but demanded a good understanding of subtle notions and relationships learned by the students independently from a textbook. On several tests more of these problems were solved in the experimental group than in the control one. On the other hand, no significant difference was found in the average number of technical (i.e., numeric or algebraic) errors. Finally, considerable change in the attitude of students following the project was observed, particularly of those less able - a certain recovery from "mathophobia," more voluntary engagement in classroom activities, and less reluctance toward learning from a textbook.

Thus the project, at least when implemented by Trelinski in his class, proved to be highly successful, and yet its dissemination was never attempted.

The experimental approach was so time-consuming that Trelinski did not dare to suggest that anybody, including himself, try to apply it in day-to-day teaching; it could only be done for a doctoral dissertation. There seems to be a rule: An educational project, to be significantly successful must also be costly, too costly for mass implementation. So most good projects die just after delivery.

CONCLUSION

From all four examples of research emanates, not by coincidence, Krygowska's pupils' strong belief that genuine mathematics can be taught to everybody, the only question being how. It is always the teaching that should be blamed for failures, not students' inability. Therefore, it is those students who fail that are the main concern of most Cracow researchers, not those who have succeeded or will succeed regardless of teaching methods and means. The impact of a didactic innovation on the average and weak student is always more important than the progress of the able student or the increase of an overall average result. If anybody wished to call this concern a principle of socialist mathematics education research, we would not object.

It is also not a coincidence that all researchers place deep trust in the observation of the teaching process as the main source of research data, at the same time that they harbor an implicit distrust of so-called educational measurements and the sophisticated statistical techniques applied to them. These attitudes are also part of Krygowska's stand on research in mathematics teaching: The subtleties of the painstaking construction of abstract mathematical structures in the pupil's mind can only be identified through close observation of his or her activity, not through collecting large numbers of written tests, which are usually used as measuring tools. If implications applicable to the classroom are to be inferred, research must be pursued mainly in the classroom, not in a laboratory. This dictum should not be perceived as a priori discrimination against certain research methods, but rather, as merely a sensible demand that adequate research methods be applied to the problems being researched.

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